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Symmetry and Multiplicity of Solutions in a Two-Dimensional Landau–de Gennes Model for Liquid Crystals

Radu Ignat, Luc Nguyen, Valeriy Slastikov &
Arghir Zarnescu

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Abstract

We consider a variational two-dimensional Landau–de Gennes model in the theory of nematic liquid crystals in a disk of radius R. We prove that under a symmetric boundary condition carrying a topological defect of degree $\frac{k}{2}$ for some given **even** non-zero integer k, there are exactly two minimizers for all large enough R. We show that the minimizers do not inherit the full symmetry structure of the energy functional and the boundary data. We further show that there are at least five symmetric critical points.

Contents

1. Introduction

The questions of symmetry and stability of critical points for the Landau–de Gennes energy functional on two dimensional domains have been recently raised in the mathematical liquid crystal community [4,15,20,23,24,27]. The particular focus of these works was on analyzing special symmetric critical points and investigating their stability properties depending on multiple parameters of the problem.

In this paper we continue the study of the symmetry, stability and multiplicity of critical points of the Landau–de Gennes energy using the same mathematical setting. The main result we establish is the uniqueness (up to reflection) of the global minimizer in the most relevant physical regime of small elastic constant under the strong anchoring boundary condition which has a topological degree $\frac{k}{2}$ with even nonzero k (see (1.7)–(1.9) below). As a consequence of this uniqueness, the minimizers satisfy a k-fold $O(2)$-symmetry (see Definition 1.1) which has not been identified earlier. Additionally, we prove the existence of two other k-fold $O(2)$-symmetric critical points which are not minimizing.

We recall the (non-dimensional) Landau–de Gennes energy functional in the disk $B_R \subset \mathbb{R}^2$ of radius $R \in (0, \infty)$ centered at the origin:

$$\mathscr{F}[Q; B_R] = \int_{B_R} \left[\frac{1}{2}|\nabla Q|^2 + f_{\text{bulk}}(Q)\right] \mathrm{d}x, \qquad Q \in H^1(B_R, \mathscr{S}_0), \quad (1.1)$$

where $\mathscr{S}_0$ is the set of Q-*tensors*:

$$\mathscr{S}_0 := \{Q \in \mathbb{R}^{3\times 3} \,:\, tr(Q) = 0,\ Q = Q^t\}. \quad (1.2)$$

The nonlinear bulk potential is given by

$$f_{\text{bulk}}(Q) = -\frac{a^2}{2}\text{tr}(Q^2) - \frac{b^2}{3}\text{tr}(Q^3) + \frac{c^2}{4}(\text{tr}(Q^2))^2 - f_*,$$

where $a^2 \geqq 0$, $b^2, c^2 > 0$ are appropriately scaled parameters and the normalizing constant f_* is chosen such that the minimum value of f_{bulk} over $\mathscr{S}_0$ is zero. A direct computation gives

$$f_* = -\frac{a^2}{3}s_+^2 - \frac{2b^2}{27}s_+^3 + \frac{c^2}{9}s_+^4 \quad (1.3)$$

with[1]

$$s_+ = \frac{b^2 + \sqrt{b^4 + 24a^2c^2}}{4c^2} > 0. \quad (1.4)$$

The set of minimizers of f_{bulk}, which we call the *limit manifold*, is given by the following set of uniaxial Q-tensors:

$$\mathscr{S}_* := \{Q \in \mathscr{S}_0 : f_{\text{bulk}}(Q) = 0\} = \left\{Q = s_+\left(v \otimes v - \frac{1}{3}I_3\right),\ \ v \in \mathbb{S}^2\right\}, \quad (1.5)$$

where I_3 is the 3×3 identity matrix.

[1] It is sometimes useful to note that the function $s \mapsto -\frac{a^2}{3}s^2 - \frac{2b^2}{27}s^3 + \frac{c^2}{9}s^4$ is minimized at $s = s_+$.

The Euler–Lagrange equations satisfied by the critical points of $\mathscr{F}$ read as

$$\Delta Q = -a^2\, Q - b^2 \left[Q^2 - \frac{1}{3}\mathrm{tr}(Q^2) I_3 \right] + c^2\, \mathrm{tr}(Q^2)\, Q \quad \text{in } B_R. \tag{1.6}$$

The Landau–de Gennes energy describes the pattern formation in liquid crystal systems, in particular, the so-called defect patterns (see for a introduction the review [30]). A well-studied limit, relating the defects in the Landau–de Gennes framework with those in the Oseen–Frank framework, is that of small elastic constant (after a suitable non-dimensionalisation—see [16]), considered, for instance, in [1,4,10,18] in two dimensional and [11,31,32] in three dimensional. Qualitative properties of defects and their stability are studied, for example, in the case of one elastic constant in two dimensional domains in [15,23,24] and in three dimensional domains in [12,22,28]. Numerical explorations of the defects in two dimensional domains and several elastic constants are available in [2,17,27].

We couple the system (1.6) with the following strong anchoring boundary condition:

$$Q(x) = Q_b(x) \text{ on } \partial B_R, \tag{1.7}$$

where the map $Q_b : \mathbb{R}^2\backslash\{0\} \to \mathscr{S}_0$ is defined, for some fixed $k \in \mathbb{Z}\backslash\{0\}$, by

$$Q_b(x) := s_+ \left(n(x) \otimes n(x) - \frac{1}{3} I_3 \right), \quad x \in \mathbb{R}^2\backslash\{0\}, \tag{1.8}$$

$$n(r\cos\varphi, r\sin\varphi) := \left(\cos\frac{k\varphi}{2}, \sin\frac{k\varphi}{2}, 0 \right), \quad r > 0,\ 0 \leqq \varphi < 2\pi. \tag{1.9}$$

Note that Q_b has image in $\{s_+ \left(v \otimes v - \frac{1}{3} I_3\right) : v \in \mathbb{S}^1\} \cong \mathbb{R}P^1$ and, as a map from $\partial B_R \cong \mathbb{S}^1$ into $\mathbb{R}P^1$ (see for instance formula (8.16) in [8]), has $\frac{1}{2}\mathbb{Z}$-valued topological degree $\frac{k}{2}$.

It is worth pointing out the difference between the cases when k is even and odd. If k is even, the vector n defined in (1.9) is continuous at $\varphi = 2\pi$, however, if k is odd there is a jump discontinuity at $\varphi = 2\pi$. Nevertheless the boundary data Q_b defined in terms of n in (1.8) is continuous for any $k \in \mathbb{Z}$, but its topological features as a map from $\partial B_R \cong \mathbb{S}^1$ into $\mathbb{R}P^1$ will depend on the parity of k, see Ball and Zarnescu [3], Bethuel and Chiron [7], Brezis, Coron and Lieb [8], Ignat and Lamy [21]. In particular, this leads to major qualitative differences in the properties of the critical points; see [3,15,23,24] for analytical studies in two dimensional domains which involve only one elastic constant, and [17,27] for numerical studies for several elastic constants. Moreover, in the limit of small elastic constant the minimal Landau–de Gennes energy becomes infinite in the case of odd k (see [10,18]) and is finite in the case of even k (see [14]). This phenomenon leads to significant differences in the structure and distribution of defects depending on the parity of k (see Appendix 4).

Two group actions on the space $H^1(B_R, \mathscr{S}_0)$. In what follows we consider two types of symmetries induced by two group actions on the space $H^1(B_R, \mathscr{S}_0)$ which keep invariant both the energy functional $\mathscr{F}$ as well as the boundary condition (1.7)–(1.9).

• k-**fold** $O(2)$-**symmetry**. For $k \in \mathbb{Z}\backslash\{0\}$, we introduce the following group action of $O(2)$ on $H^1(B_R, \mathscr{S}_0)$. We identify $O(2) \sim \{0, 1\} \times \mathbb{S}^1 \sim \{0, 1\} \times [0, 2\pi)$ and define the action of $O(2)$ on $H^1(B_R, \mathscr{S}_0)$ by

$$(\alpha, \psi, Q) \in \{0, 1\} \times [0, 2\pi) \times H^1(B_R, \mathscr{S}_0) \mapsto Q_{\alpha,\psi} \in H^1(B_R, \mathscr{S}_0). \tag{1.10}$$

Here $Q_{\alpha,\psi}$ is defined as

$$Q_{\alpha,\psi}(x) := L^\alpha \mathcal{R}_k^t(\psi) Q\Big(P_2\big(L^\alpha \mathcal{R}_2(\psi)\tilde{x}\big)\Big)\mathcal{R}_k(\psi)L^\alpha \\ \text{for almost every } x = (x_1, x_2) \in B_R, \tag{1.11}$$

with $\tilde{x} = (x_1, x_2, 0)$, $P_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $P_2(x_1, x_2, x_3) = (x_1, x_2)$,

$$\mathcal{R}_k(\psi) := \begin{pmatrix} \cos\left(\frac{k}{2}\psi\right) & -\sin\left(\frac{k}{2}\psi\right) & 0 \\ \sin\left(\frac{k}{2}\psi\right) & \cos\left(\frac{k}{2}\psi\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.12}$$

representing an in-plane rotation about $e_3 = (0, 0, 1)$ by angle $\frac{k}{2}\psi$, and

$$L := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.13}$$

defining the reflection with respect to the plane perpendicular to the $(0, 1, 0)$-direction.

Definition 1.1. Let $k \in \mathbb{Z}\backslash\{0\}$. The subset of $H^1(B_R, \mathscr{S}_0)$ that is invariant under the group action (1.10) is called the set of k-**fold** $O(2)$-**symmetric** maps. Such a map $Q \in H^1(B_R, \mathscr{S}_0)$ is therefore characterized by

$$Q = Q_{\alpha,\psi} \quad \text{in } B_R \text{ for every } (\alpha, \psi) \in \{0, 1\} \times [0, 2\pi). \tag{1.14}$$

Sometimes when k is clear (uniquely determined) from the context, we will omit "k-fold" and simply call the above property as $O(2)$-symmetry. The following proposition provides a characterization of k-fold $O(2)$-symmetric maps in the case of even k (its proof is postponed until Section 2):

Proposition 1.2. *Let $k \in 2\mathbb{Z}\backslash\{0\}$. A map $Q \in H^1(B_R, \mathscr{S}_0)$ is k-fold $O(2)$-symmetric if and only if*

$$Q(x) = w_0(r)E_0 + w_1(r)E_1 + w_3(r)E_3 \\ \textit{for almost every } x = (r\cos\varphi, r\sin\varphi) \in B_R, \tag{1.15}$$

where

$$E_0 = \sqrt{\frac{3}{2}}\left(e_3 \otimes e_3 - \frac{1}{3}I_3\right), E_1 = \sqrt{2}\left(n \otimes n - \frac{1}{2}I_2\right),$$

$$E_3 = \frac{1}{\sqrt{2}}(n \otimes e_3 + e_3 \otimes n) \tag{1.16}$$

with n given by (1.9)*,* $e_3 = (0, 0, 1)$*,* $I_2 = I_3 - e_3 \otimes e_3$*,* $w_0 \in H^1((0, R); r\,\mathrm{d}r)$ *and* $w_1, w_3 \in H^1((0, R); r\,\mathrm{d}r) \cap L^2((0, R); \frac{1}{r}\,\mathrm{d}r)$.

When k is odd, k-fold $O(2)$-symmetric maps are of the form $Q(x) = w_0(r)E_0 + w_1(r)E_1$, that is, $w_3 = 0$ in (1.15). See Remark 2.7.

• $\mathbb{Z}_2$-**symmetry**. We introduce the group action of $\mathbb{Z}_2$ on $H^1(B_R, \mathscr{S}_0)$:

$$(\alpha, Q) \in \mathbb{Z}_2 \times H^1(B_R, \mathscr{S}_0) \mapsto J^\alpha Q J^\alpha \in H^1(B_R, \mathscr{S}_0), \tag{1.17}$$

where J stands for the reflection with respect to the plane perpendicular to the $(0, 0, 1)$-direction.:

$$J := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.18}$$

Definition 1.3. The subset of $H^1(B_R, \mathscr{S}_0)$ that is invariant under the group action (1.17) is called the set of $\mathbb{Z}_2$-**symmetric** maps. Such a map $Q \in H^1(B_R, \mathscr{S}_0)$ is therefore characterized by

$$Q = JQJ \quad \text{in } B_R.$$

We will see in Proposition 2.8 that a map $Q \in H^1(B_R, \mathscr{S}_0)$ is $\mathbb{Z}_2$-symmetric if and only if $e_3 = (0, 0, 1)$ is an eigenvector of $Q(x)$ for almost all $x \in B_R$. We note an important difference between the definitions of k-fold $O(2)$-symmetry and $\mathbb{Z}_2$-symmetry: the $O(2)$-action on $H^1(B_R, \mathscr{S}_0)$ applies to both the domain and the target space while the $\mathbb{Z}_2$-action applies only to the target space.

It is clear that if Q is a minimizer (or a critical point) of $\mathscr{F}$ under the boundary condition (1.7) then the elements of its orbit under the k-fold $O(2)$-action as well as the $\mathbb{Z}_2$-action are also minimizers (or critical points, respectively). A natural question therefore arises: do minimizers/critical points of $\mathscr{F}$ (under (1.7)) have k-fold $O(2)$-symmetry, or $\mathbb{Z}_2$-symmetry, or both, or maybe none? Some partial answers are available in the literature. In a work of Bauman, Park and Phillips [4], which is not directly related to symmetry issues, it was shown that, for $|k| \neq 0, 1$ and as $R \to \infty$, there exist none-$O(2)$-symmetric critical points. Their results might tempt one to extrapolate a lack of symmetry in general. This intuition would be also apparently supported by the numerical simulations in Hu, Qu and Zhang [20] which observed lack of symmetry for a certain radius. However, in [27] the k-fold $O(2)$-symmetry was numerically observed for a minimizer in the case of even k and large enough radius R (probably larger than in the examples explored numerically in [20]).

Definition 1.4. For $k \in \mathbb{Z}\backslash\{0\}$, a map $Q \in H^1(B_R, \mathscr{S}_0)$ is called (k**-fold**) $\mathbb{Z}_2 \times O(2)$-**symmetric** if Q is both (k-fold) $O(2)$-symmetric and $\mathbb{Z}_2$-symmetric.

We will see later (in Section 2) that all $\mathbb{Z}_2 \times O(2)$-symmetric maps are of the form

$$Q(x) = w_0(|x|)E_0 + w_1(|x|)E_1 \text{ for almost every } x \in B_R.^{2} \tag{1.19}$$

It is known from [23] that all $\mathbb{Z}_2 \times O(2)$-symmetric critical points of $\mathscr{F}$ coincide with the so-called k-radially symmetric critical points; see Section 2 for more details. Note that the boundary data Q_b defined in (1.8) is $\mathbb{Z}_2 \times O(2)$-symmetric on ∂B_R. However, we will prove that the minimizers of $\mathscr{F}[\cdot; B_R]$ under the boundary condition (1.8) do not satisfy this symmetry (namely they are not $\mathbb{Z}_2$-symmetric).

The structure and stability properties of $\mathbb{Z}_2 \times O(2)$-symmetric critical points were investigated in [15,23,24]. In particular, it was proved that

- when $b = 0$ and $R < \infty$ they are minimizers of the Landau–de Gennes energy for all $k \in \mathbb{Z}\setminus\{0\}$ (see [15]);
- when $b \neq 0$, $\mathbb{N} \ni |k| > 1$ and R is large enough they are unstable (see [23]);
- when $b \neq 0$ and $k = \pm 1$ they are locally stable for all $R \leqq \infty$ under suitable condition on w_0 and w_1 in (1.19) (see [24]).[3]

In this paper we focus on the case

$$k \in 2\mathbb{Z}\setminus\{0\},$$

where the k-fold $O(2)$-symmetry does not imply in general the $\mathbb{Z}_2$-symmetry (some remarks on the case k odd are provided in Appendix 4). Our main result states that for large enough radius R the Landau–de Gennes energy (1.1) under the boundary condition (1.7) has exactly two minimizers and these minimizers are k-fold $O(2)$-symmetric and $\mathbb{Z}_2$-conjugate to each other.

Theorem 1.5. *Let $a^2 \geqq 0$, $b^2, c^2 > 0$ be any fixed constants and $k \in 2\mathbb{Z}\setminus\{0\}$. There exists some $R_0 = R_0(a^2, b^2, c^2, k) > 0$ such that for all $R > R_0$, there exist exactly two global minimizers $Q_R^{\pm}$ of $\mathscr{F}[\cdot; B_R]$ subjected to the boundary condition (1.7) and these minimizers are k-fold $O(2)$-symmetric (but not $\mathbb{Z}_2 \times O(2)$-symmetric). The minimizers $Q_R^{\pm}$ are $\mathbb{Z}_2$-conjugate to one another, namely, $Q_R^{\pm} = JQ_R^{\mp}J \neq Q_R^{\mp}$ and have the form*

$$Q_R^{\pm}(x) = w_0(|x|)E_0 + w_1(|x|)E_1 \pm w_3(|x|)E_3 \quad \textit{for every } x \in B_R, \tag{1.20}$$

where E_0, E_1 and E_3 are given by (1.16) and $w_3 > 0$ in $(0, R)$.

It is clear that the Euler–Lagrange equation (1.6) for $Q_R^{\pm}$ then reduces to a system of ODEs for $(w_0, w_1, 0, \pm w_3, 0)$ with the boundary condition $w_0(R) = -\frac{s_+}{\sqrt{6}}$, $w_1(R) = \frac{s_+}{\sqrt{2}}$ and $w_3(R) = 0$. See Remark 2.4.

The idea of the proof of Theorem 1.5 is presented in Section 3.1. An assumption in the above theorem concerns the radius of the domain which is taken to be large

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[2] In particular, in view of Remark 2.7, if k is odd, all k-fold $O(2)$-symmetric maps are $\mathbb{Z}_2 \times O(2)$-symmetric.

[3] For $b^4 \leqq 3a^2c^2$, the condition reduces to $w_0 < 0$ and $w_1 > 0$.

enough. This is a physically relevant assumption, capturing the most interesting physical regime of small elastic constant (as explained in [16] and studied, for instance, in [1,4,10,14,18] in two dimensional and [11,31,32] in three dimensional).

We would like to draw attention to our related uniqueness results in a Ginzburg–Landau settings [25,26] where the bulk potential satisfies a suitable global convexity assumption. In these articles, we established a link between the so-called non-escaping phenomenon and uniqueness of minimizers. In the context of Q-tensors, a non-escaping phenomenon would mean the existence of $O(2)$-symmetric critical point Q such that $Q \cdot E_3$ does not change sign. While it is not hard to prove the existence of such critical points for large R (see the last paragraph in the proof of Theorem 3.1), the method in [25,26] does not apply to the present setting as our bulk potential f_{bulk} does not satisfy the relevant global convexity. In a sequel to the present article, we will apply the method developed here to prove a similar uniqueness result for minimizers of a Ginzburg–Landau type energy functional where the bulk potential satisfies only a local convexity property near the limit manifold.

Our second result concerns the multiplicity of k-fold $O(2)$-symmetric critical points of $\mathscr{F}$. This is coherent with the numerical simulations in [27, Section 3.2] for $k = 2$ and [20, Section 2.2], which observed, for large enough R, that there can be several distinct solutions, corresponding to boundary conditions (1.7).

Theorem 1.6. *Let $a^2 \geqq 0, b^2, c^2 > 0$ be any fixed constants and $k \in 2\mathbb{Z}\backslash\{0\}$. There exists some $R_1 = R_1(a^2, b^2, c^2, k) > 0$ such that for all $R > R_1$, there exist at least five k-fold $O(2)$-symmetric critical points of $\mathscr{F}[\cdot; B_R]$ subjected to the boundary condition (1.7). At least four of these solutions are* **not** *$\mathbb{Z}_2 \times O(2)$-symmetric.*

The rough idea of proving Theorem 1.6 is the following: Theorem 1.5 gives us two global minimizers $Q_R^{\pm}$. By the mountain pass theorem, there is a mountain pass critical point, denoted Q_R^{mp} that connects these two ($\mathbb{Z}_2$-conjugate) minimizers $Q_R^{\pm}$. The main point in the proof of Theorem 1.6 is to show that the mountain pass solution Q_R^{mp} does not coincide with the k-radially symmetric critical point Q_R^{str} constructed in [23]. This is done by an energy estimate showing in particular the existence of paths between $Q_R^{\pm}$ for which the energy is uniformly bounded with respect to R, see (4.6). As the maps $Q_R^{\pm}$, after suitably rescaled, converge to two $\mathscr{S}_*$-valued minimizing harmonic maps of different topological nature, this highlights the difficulty of constructing that path; see Section 4 for a more detailed discussion. Moreover, we show that the mountain pass critical point is not $\mathbb{Z}_2$-symmetric, thus its $\mathbb{Z}_2$-conjugate $\tilde{Q}_R^{mp}$ is also a critical point, thus yielding five different critical points.

In Table 1, we summarize the properties of the critical points from Theorem 1.6.[4]

The two global minimizers $Q_R^{\pm}$ as well as the two mountain pass critical points Q_R^{mp} and $\tilde{Q}_R^{mp}$ are k-fold $O(2)$-symmetric but not $\mathbb{Z}_2$-symmetric. The map Q_R^{str} is a minimizer among $\mathbb{Z}_2 \times O(2)$-symmetric maps.[5] The subscript k in the little

[4] See Appendix 4 for related remarks regarding the case when k is odd.

[5] It is an open problem if the w_0 and w_1 components of Q_R^{str} satisfy $w_0 < 0$ and $w_1 > 0$. See [23, Open problem 3.2].

Table 1. Properties of critical points for even $k \neq 0$ and large radius R

Critical point	Stability	w_3 in (1.15)	Symmetry	Energy as $R \to \infty$
$Q_R^{\pm}$	Yes	$w_3^+ = -w_3^- > 0$	$O(2)$-symmetry	$4\pi s_+^2 \ \|k\| + o_k(1)$
Q_R^{str}	No	$w_3 \equiv 0$	$\mathbb{Z}_2 \times O(2)$-symmetry	$\frac{\pi k^2 s_+^2}{2} \ln R + o_k(\ln R)$
Q_R^{mp} and $\tilde{Q}_R^{mp}$	No	$w_3 = -\tilde{w}_3 \not\equiv 0$	$O(2)$-symmetry	$O_k(1)$

o-terms indicates that the rate of convergence may depend on k. The subscript k in the big O-terms indicates that the implicit constant may depend on k. The energy of $Q_R^{\pm}$ is bounded from above by and converges as $R \to \infty$ to the Dirichlet energy of the $\mathscr{S}_*$-valued minimal harmonic map(s) on B_1, which is $4\pi s_+^2 |k|$; see (4.7). The asymptotic behavior of the energy of Q_R^{str} as $R \to \infty$ is proved in Lemma 4.1. The estimate for the energy of the mountain pass solutions is given in (4.5). In addition to these solutions, we also have the non-$O(2)$-symmetric solutions constructed in [4], which have energy $O(|k| \ln R)$ for large R, see [4, Theorem B].

To dispel confusion, we note that the Ginzburg–Landau counterpart for our model is the two dimensional–three dimensional Ginzburg–Landau model (see [26, Theorem 1.1]). In particular, the minimal energy remains bounded as $R \to \infty$, which is contrary to the two dimensional–two dimensional Ginzburg–Landau case where the minimal energy grows like $\ln R$ as $R \to \infty$ (see for example the seminal book of BÉTHUEL, BREZIS AND HÉLEIN [6] or [29,33]). In the two dimensional–three dimensional case, it was shown in [26] that, for every $k \in 2\mathbb{Z}\backslash\{0\}$ and under the boundary condition (1.9), there exists $R_* > 0$ such that the Ginzburg–Landau energy functional has a unique critical point for $R \leqq R_*$ and has exactly two minimizers which 'escape in the third dimension' for $R > R_*$.

The paper is organized as follows: in Section 2 we present some basic facts about the two types of symmetry induced by the $O(2)$- and $\mathbb{Z}_2$-group actions, and, in particular, about k-fold $SO(2)$-symmetric minimizers of the Landau–de Gennes energy. Section 3 contains the main part of the paper, namely, the proof of Theorem 1.5. The overall idea and main mathematical set-up of the proof are described in the Sections 3.1–3.3. Sections 3.4–3.7 contain formulations and proofs of the auxiliary results used in Sections 3.8–3.9 to prove Theorem 1.5. In Section 4 we prove the existence of multiple critical points for large enough domains, namely Theorem 1.6. In Appendix 4 we provide a couple of remarks on the minimal energy and the symmetry properties of minimizers of $\mathscr{F}[\cdot; B_R]$ for odd k. Finally, in the Appendices 4, 4, 4 we put some technical details required to prove our results.

2. Structure of Symmetric Maps: Proof of Proposition 1.2

We work with a moving (that is, x-dependent) orthonormal basis of the space $\mathscr{S}_0$ (defined in (1.2)), which is compatible with the boundary condition (1.7). We use polar coordinates in $\mathbb{R}^2$, that is, $x = (r \cos\varphi, r \sin\varphi)$ with $r > 0$ and $\varphi \in [0, 2\pi)$.

Let $\{e_i\}_{i=1}^3$ be the standard basis of $\mathbb{R}^3$, and let

$$n(x) = \left(\cos\frac{k\varphi}{2}, \sin\frac{k\varphi}{2}, 0\right), \quad m(x) = \left(-\sin\frac{k\varphi}{2}, \cos\frac{k\varphi}{2}, 0\right), \\ x \in \mathbb{R}^2. \tag{2.1}$$

We endow $\mathscr{S}_0$ with the Frobenius scalar product of symmetric matrices $Q \cdot P = \operatorname{tr}(Q\,P)$ and the induced norm $|Q| = (Q \cdot Q)^{1/2}$. We define, for $x \in \mathbb{R}^2$, the following orthonormal basis of $\mathscr{S}_0$:

$$\begin{aligned} E_0 &= \sqrt{\frac{3}{2}}\left(e_3 \otimes e_3 - \frac{1}{3}I_3\right), \quad E_1 = \sqrt{2}\left(n \otimes n - \frac{1}{2}I_2\right), \\ E_2 &= \frac{1}{\sqrt{2}}(n \otimes m + m \otimes n), \\ E_3 &= \frac{1}{\sqrt{2}}(n \otimes e_3 + e_3 \otimes n), \ E_4 = \frac{1}{\sqrt{2}}(m \otimes e_3 + e_3 \otimes m). \end{aligned} \tag{2.2}$$

Recall that I_3 is the 3×3 identity matrix and $I_2 = I_3 - e_3 \otimes e_3$. It should be noted that this choice of basis elements for $\mathscr{S}_0$ differs slightly from [23,24] where both even and odd values of k were considered. This is due to the fact that E_3 and E_4 are continuous when we identify $\varphi = 0$ with $\varphi = 2\pi$ if and only if k is even.

We identify a map $Q : B_R \to \mathscr{S}_0$ with a map $\mathbf{w} = (w_0, \ldots, w_4) : B_R \to \mathbb{R}^5$ via $Q = \sum_{i=0}^4 w_i\, E_i$. Then $|Q|^2 = |\mathbf{w}|^2$,

$$\begin{aligned} \operatorname{tr}(Q^3) &= \frac{\sqrt{6}}{12}\big[2w_0^3 - 6w_0(w_1^2 + w_2^2) + 3w_0(w_3^2 + w_4^2) \\ &\quad + 3\sqrt{3}w_1(w_3^2 - w_4^2) + 6\sqrt{3}w_2 w_3 w_4\big], \\ |\nabla Q|^2 &= |\partial_r \mathbf{w}|^2 + \frac{1}{r^2}\Big[|\partial_\varphi w_0|^2 + |\partial_\varphi w_1 - k w_2|^2 + |\partial_\varphi w_2 + k w_1|^2 \\ &\quad + |\partial_\varphi w_3 - \frac{k}{2}w_4|^2 + |\partial_\varphi w_4 + \frac{k}{2}w_3|^2\Big], \end{aligned} \tag{2.3}$$

where we have used the following identities for even k:

$$\partial_\varphi E_1 = kE_2, \quad \partial_\varphi E_2 = -kE_1, \quad \partial_\varphi E_3 = \frac{k}{2}E_4, \quad \partial_\varphi E_4 = -\frac{k}{2}E_3.$$

The Landau–de Gennes energy (1.1) becomes

$$\begin{aligned} \mathscr{F}[Q; B_R] &= I[\mathbf{w}] \\ &:= \int_{B_R} \Big\{\frac{1}{2}|\partial_r \mathbf{w}|^2 + \frac{1}{2r^2}\Big[|\partial_\varphi w_0|^2 + |\partial_\varphi w_1 - k w_2|^2 + |\partial_\varphi w_2 + k w_1|^2 \\ &\quad + |\partial_\varphi w_3 - \frac{k}{2}w_4|^2 + |\partial_\varphi w_4 + \frac{k}{2}w_3|^2\Big] \\ &\quad + \left(-\frac{a^2}{2} + \frac{c^2}{4}|\mathbf{w}|^2\right)|\mathbf{w}|^2 \end{aligned}$$

$$- \frac{b^2\sqrt{6}}{36}\big[2w_0^3 - 6w_0(w_1^2 + w_2^2) + 3w_0(w_3^2 + w_4^2)$$
$$+ 3\sqrt{3}w_1(w_3^2 - w_4^2) + 6\sqrt{3}w_2w_3w_4\big] - f_*\Big\}r\,\mathrm{d}r\,d\varphi. \tag{2.4}$$

The boundary condition (1.7) becomes

$$\mathbf{w}(x) = \left(-\frac{s_+}{\sqrt{6}}, \frac{s_+}{\sqrt{2}}, 0, 0, 0\right) \text{ on } \partial B_R. \tag{2.5}$$

In the introduction, we defined a group action of $O(2)$ on $H^1(B_R, \mathscr{S}_0)$. There we viewed $O(2)$ as a direct product of $\{0, 1\}$ and $SO(2)$. This naturally induces two group actions of $\{0, 1\} \cong \mathbb{Z}_2$ and of $SO(2)$, as subgroups of $O(2)$, on $H^1(B_R, \mathscr{S}_0)$.

Definition 2.1. Let $k \in \mathbb{Z}\backslash\{0\}$. A map $Q \in H^1(B_R, \mathscr{S}_0)$ is said to be $\{0, 1\}$-**symmetric** if

$$Q = Q_{1,0} \quad \text{in } B_R. \tag{2.6}$$

A map $Q \in H^1(B_R, \mathscr{S}_0)$ is said to be k-**fold** $SO(2)$-**symmetric** if

$$Q = Q_{0,\psi} \quad \text{in } B_R \text{ for every } \psi \in [0, 2\pi). \tag{2.7}$$

Here $Q_{\alpha,\psi}$ is defined by (1.11).

Note that the groups $\mathbb{Z}_2$ and $\{0, 1\}$ are isomorphic, but we have deliberately distinguished the notations to avoid confusion with the $\mathbb{Z}_2$-action defined in the introduction. Moreover, the nature of the two group actions are somewhat different. The $\mathbb{Z}_2$-action is related to the reflection along e_3 direction of the target, while the $\{0, 1\}$-action is related to the reflection along the e_2 direction in both the domain and the target.

Definition 2.2. [23, Definition 1.1]. Let $k \in \mathbb{Z}\backslash\{0\}$. A map $Q \in H^1(B_R, \mathscr{S}_0)$ is said to be k-**radially symmetric** (or equivalently $\mathbb{Z}_2 \times SO(2)$-**symmetric**) if Q is $\mathbb{Z}_2$-symmetric and k-fold $SO(2)$-symmetric.

The k-fold $SO(2)$-symmetry is exactly condition (**H2**) in [23, Definition 1.1]. We have the following characterization:

Proposition 2.3. *Let $R \in (0, \infty]$ and $k \in 2\mathbb{Z}\backslash\{0\}$. A map $Q \in H^1(B_R, \mathscr{S}_0)$ is k-fold $SO(2)$-symmetric if and only if it can be represented as*

$$Q(x) = \sum_{i=0}^{4} w_i(r)\, E_i \textit{ for almost every } x = r(\cos\varphi, \sin\varphi) \in B_R,$$

where E_i's are given by (2.2), $w_i = Q \cdot E_i$, $w_0 \in H^1((0, R); r\,\mathrm{d}r)$ and $w_1, w_2, w_3, w_4 \in H^1((0, R); r\,\mathrm{d}r) \cap L^2((0, R); \frac{1}{r}\,\mathrm{d}r)$.

Proof. Suppose that $Q \in H^1(B_R, \mathscr{S}_0)$ is k-fold $SO(2)$-symmetric . By [23, Proposition 2.1], there exist $w_0 \in H^1((0,R); r\,\mathrm{d}r)$ and $w_1, w_2, \tilde{w}, \hat{w} \in H^1((0,R); r\,\mathrm{d}r) \cap L^2((0,R); \frac{1}{r}\,\mathrm{d}r)$ such that

$$\begin{aligned} Q(x) &= \sum_{i=0}^{2} w_i(r)\,E_i + \left(\tilde{w}(r)\cos\frac{k}{2}\varphi + \hat{w}(r)\sin\frac{k}{2}\varphi\right)\frac{1}{\sqrt{2}}(e_1\otimes e_3 + e_3\otimes e_1) \\ &\quad + \left(-\hat{w}(r)\cos\frac{k}{2}\varphi + \tilde{w}(r)\sin\frac{k}{2}\varphi\right)\frac{1}{\sqrt{2}}(e_2\otimes e_3 + e_3\otimes e_2) \\ &= \sum_{i=0}^{2} w_i(r)\,E_i + \tilde{w}(r)E_3 - \hat{w}(r)E_4 \text{ for almost every } x \in B_R. \end{aligned}$$

This gives the desired representation.

Consider the converse. Suppose that $Q(x) = \sum_{i=0}^{4} w_i(r)E_i$. A direct check shows that the basis elements $E_0, \ldots, E_4$ are k-fold $SO(2)$-symmetric. Hence Q is k-fold $SO(2)$-symmetric. If we have further that $w_0 \in H^1((0,R); r\,\mathrm{d}r)$ and $w_1, \ldots, w_4 \in H^1((0,R); r\,\mathrm{d}r) \cap L^2((0,R); \frac{1}{r}\,\mathrm{d}r)$, then, by (2.3), $|\nabla Q|$ is square integrable over B_R. It follows that $Q \in H^1(B_R, \mathscr{S}_0)$. □

Remark 2.4. Let Q be a $SO(2)$-symmetric map. Q is a critical point of $\mathscr{F}$ if and only if its components $w_0, \ldots, w_4$ satisfy

$$w_0'' + \frac{1}{r}w_0' = w_0\Big(-a^2 + c^2|\mathbf{w}|^2 - \frac{b^2}{\sqrt{6}}w_0\Big) + \frac{b^2}{\sqrt{6}}(w_1^2 + w_2^2) - \frac{b^2}{2\sqrt{6}}(w_3^2 + w_4^2), \tag{2.8}$$

$$w_1'' + \frac{1}{r}w_1' - \frac{k^2}{r^2}\,w_1 = w_1\Big(-a^2 + c^2|\mathbf{w}|^2 + \frac{2b^2}{\sqrt{6}}w_0\Big) - \frac{b^2}{2\sqrt{2}}(w_3^2 - w_4^2), \tag{2.9}$$

$$w_2'' + \frac{1}{r}w_2' - \frac{k^2}{r^2}\,w_2 = w_2\Big(-a^2 + c^2|\mathbf{w}|^2 + \frac{2b^2}{\sqrt{6}}w_0\Big) - \frac{b^2}{\sqrt{2}}w_3\,w_4, \tag{2.10}$$

$$w_3'' + \frac{1}{r}w_3' - \frac{k^2}{4r^2}\,w_3 = w_3\Big(-a^2 + c^2|\mathbf{w}|^2 - \frac{b^2}{\sqrt{6}}w_0 - \frac{b^2}{\sqrt{2}}w_1\Big) - \frac{b^2}{\sqrt{2}}w_2\,w_4, \tag{2.11}$$

$$w_4'' + \frac{1}{r}w_4' - \frac{k^2}{4r^2}\,w_4 = w_4\Big(-a^2 + c^2|\mathbf{w}|^2 - \frac{b^2}{\sqrt{6}}w_0 + \frac{b^2}{\sqrt{2}}w_1\Big) - \frac{b^2}{\sqrt{2}}w_2\,w_3. \tag{2.12}$$

On the other hand, the $\{0,1\}$-symmetry imposes that the w_2 and w_4 components are odd in the x_2 variable. More precisely we have the following:

Proposition 2.5. *Let $R \in (0, \infty]$ and suppose that $Q \in H^1(B_R, \mathscr{S}_0)$. Then Q is $\{0, 1\}$-symmetric if and only if*

$$w_i(x_1, -x_2) = -w_i(x_1, x_2), \text{ for almost every } (x_1, x_2) \in B_R, i \in \{2, 4\} \tag{2.13}$$

where $w_i = Q \cdot E_i$ and the E_i's are defined in (2.2).

Proof. The $\{0, 1\}$-symmetry is equivalent to $Q(x) = LQ(P_2(L\tilde{x}))L$ for almost everywhere $x \in B_R$, where P_2, L and $\tilde{x}$ are as in (1.11). This means that

$$\sum_{i=0}^{4} w_i(x_1, x_2)E_i(x_1, x_2) = \sum_{i=0}^{4} w_i(x_1, -x_2)LE_i(x_1, -x_2)L.$$

Noting that $Lm(x_1, -x_2) = -m(x_1, x_2)$, $Ln(x_1, -x_2) = n(x_1, x_2)$,

$$LE_i(x_1, -x_2)L = E_i(x_1, x_2) \text{ for } i \in \{0, 1, 3\},$$
$$LE_j(x_1, -x_2)L = -E_j(x_1, x_2) \text{ for } j \in \{2, 4\},$$

we obtain the conclusion. □

Proof of Proposition 1.2. The k-fold $SO(2)$-symmetry implies, by the Proposition 2.3, that the components $w_i, i \in \{0, \dots, 4\}$ are radial, hence, in particular, we have $w_2(x_1, -x_2) = w_2(x_1, x_2)$ and $w_4(x_1, -x_2) = w_4(x_1, x_2)$. Since we also have $\{0, 1\}$-symmetry, the relations (2.13) hold, which lead to $w_2 = w_4 \equiv 0$, as claimed. □

Remark 2.6. We recall [23, Corollary 2.2] that, for $k \in \mathbb{Z}\backslash\{0\}$, k-radially symmetric maps (that is, k-fold $\mathbb{Z}_2 \times SO(2)$-symmetric maps) are of the form

$$Q(x) = w_0(r)E_0 + w_1(r)E_1 + w_2(r)E_2.$$

For $k \in 2\mathbb{Z}\backslash\{0\}$, note the difference between the non-zero components of $O(2)$-symmetric maps (that is, $\{0, 1\} \times SO(2)$-symmetric maps) and k-radially symmetric maps: $O(2)$-symmetric maps have components w_0, w_1 and w_3 while k-radially symmetric maps have components w_0, w_1 and w_2.

Remark 2.7. Let $R \in (0, \infty]$, k be an odd integer and $Q \in H^1(B_R, \mathscr{S}_0)$. Then Q is k-fold $O(2)$-symmetric if and only if

$$Q(x) = w_0(r)E_0 + w_1(r)E_1$$

where $w_0 \in H^1((0, R); r\,\mathrm{d}r)$ and $w_1 \in H^1((0, R); r\,\mathrm{d}r) \cap L^2((0, R); \frac{1}{r}\,\mathrm{d}r)$. Indeed, by [23, Proposition 2.1], the k-fold $SO(2)$-symmetry implies $Q = w_0(r)E_0 + w_1(r)E_1 + w_2(r)E_2$. Then $\{0, 1\}$-symmetry implies by (2.13) that $w_2 \equiv 0$. The converse is clear (cf. (2.3)).

We also have the following characterization of $\mathbb{Z}_2$-symmetry:

Proposition 2.8. *Let $R \in (0, \infty]$ and suppose that $Q \in H^1(B_R, \mathscr{S}_0)$. Then the following conditions are equivalent:*

1. *Q is $\mathbb{Z}_2$-symmetric.*
2. *$e_3 = (0, 0, 1)$ is an eigenvector of $Q(x)$ for almost all $x \in B_R$.*[6]
3. *$Q(x) = \sum_{i=0}^{2} w_i(x) E_i$ for almost all $x \in B_R$.*

Proof. We know that any $Q \in H^1(B_R, \mathscr{S}_0)$ can be represented as $Q(x) = \sum_{i=0}^{4} w_i(x) E_i$.
Step 1. We prove ($1 \implies 2$). Suppose Q is $\mathbb{Z}_2$-symmetric. Then

$$\begin{aligned} Q(x) &= \sum_{i=0}^{4} w_i(x) E_i = J Q(x) J = \sum_{i=0}^{4} w_i(x) J E_i J \\ &= \sum_{i=0}^{2} w_i(x) E_i - w_3(x) E_3 - w_4(x) E_4. \end{aligned}$$

Therefore we obtain $w_3 = w_4 = 0$, $Q(x) = \sum_{i=0}^{2} w_i(x) E_i$ and hence e_3 is an eigenvector of $Q(x)$ for almost every $x \in B_R$.
Step 2. We prove ($2 \implies 3$). Assume now that e_3 is an eigenvector of $Q(x)$ for almost every $x \in B_R$. Therefore there exists $\lambda(x)$ such that

$$\lambda(x) e_3 = Q(x) e_3 = \sum_{i=0}^{4} w_i(x) E_i e_3 = \sqrt{\frac{2}{3}} w_0(x) e_3 + \frac{1}{\sqrt{2}} w_3(x) n + \frac{1}{\sqrt{2}} w_4(x) m.$$

Since e_3, n and m form an orthonormal basis of $\mathbb{R}^3$, it is clear that $w_3(x) = w_4(x) = 0$ almost every $x \in B_R$.
Step 3. We prove ($3 \implies 1$). Assume that $Q(x) = \sum_{i=0}^{2} w_i(x) E_i$ then it is straightforward to check that $Q = JQJ$, that is Q is $\mathbb{Z}_2$-symmetric. □

We now give a characterization of $\mathbb{Z}_2 \times O(2)$-symmetric maps.

Proposition 2.9. *Let $R \in (0, \infty]$ and $k \in 2\mathbb{Z} \backslash \{0\}$. A map $Q \in H^1(B_R, \mathscr{S}_0)$ is $\mathbb{Z}_2 \times O(2)$-symmetric if and only if*

$$Q(x) = w_0(r) E_0 + w_1(r) E_1 \textit{ for almost every } x = r(\cos\varphi, \sin\varphi) \in B_R,$$

where $w_0 \in H^1((0, R); r\,\mathrm{d}r)$ and $w_1 \in H^1((0, R); r\,\mathrm{d}r) \cap L^2((0, R); \frac{1}{r}\,\mathrm{d}r)$.

Proof. By Proposition 1.2, the $O(2)$-symmetry implies

$$Q(x) = w_0(r) E_0 + w_1(r) E_1 + w_3(r) E_3,$$

where $w_0 \in H^1((0, R); r\,\mathrm{d}r)$ and $w_1, w_3 \in H^1((0, R); r\,\mathrm{d}r) \cap L^2((0, R); \frac{1}{r}\,\mathrm{d}r)$. By Proposition 2.8 the $\mathbb{Z}_2$-symmetry implies that $w_3 \equiv 0$. The converse is clear. □

[6] This is the assumption **(H1)** in [23, Definition 1.1].

For $k \in 2\mathbb{Z}\backslash\{0\}$, we next note a connection between $SO(2)$-symmetric and $O(2)$-symmetric minimizers: Under an $O(2)$-symmetric boundary condition, in particular (1.7), $SO(2)$-symmetric minimizers of $\mathscr{F}$ are in fact $O(2)$-symmetric. We do not know however if this remains true for all $SO(2)$-symmetric critical points. See [23, Proposition 1.3 and Remark 1.4] for a related statement that $\mathbb{Z}_2 \times SO(2)$ critical points are in fact $\mathbb{Z}_2 \times O(2)$-symmetric.

Proposition 2.10. *Let $R \in (0, \infty)$ and $k \in 2\mathbb{Z}\backslash\{0\}$. If $Q \in H^1(B_R, \mathscr{S}_0)$ is a minimizer of $\mathscr{F}[\cdot; B_R]$ in the set of all k-fold $SO(2)$-symmetric Q tensors satisfying an $O(2)$-symmetric boundary condition, then Q satisfies* (1.6) *and*

$$Q(x) = w_0(|x|)\, E_0 + w_1(|x|)\, E_1 + w_3(|x|)\, E_3 \text{ for all } x \in B_R,$$

that is Q is $O(2)$-symmetric. In addition, if the boundary data is $\mathbb{Z}_2 \times O(2)$-symmetric,[7] *then $\tilde{Q} = JQJ = w_0\, E_0 + w_1\, E_1 - w_3\, E_3$ is also a minimizer of $\mathscr{F}[\cdot; B_R]$ in the same set of competitors.*

Proof. Write $Q(x) = \sum_{i=0}^{4} w_i(r)\, E_i$ as in Proposition 2.3 and let $\mathbf{w} = (w_0, \ldots, w_4)$. We will show that $w_2 \equiv w_4 \equiv 0$.

We will only consider the case where $w_1(R) \geqq 0$ and $w_3(R) \geqq 0$. (Note that the $O(2)$-symmetry of the boundary data implies that $w_2(R) = w_4(R) = 0$.) The other cases are treated similarly.

Observe that

$$\begin{aligned} w_1(w_3^2 - w_4^2) + 2w_2w_3w_4 &\leqq \sqrt{w_1^2 + w_2^2}\sqrt{(w_3^2 - w_4^2)^2 + (2w_3w_4)^2} \\ &= \sqrt{w_1^2 + w_2^2}(w_3^2 + w_4^2). \end{aligned}$$

Therefore, by (2.4),

$$I[\mathbf{w}] \geqq I\left[w_0, \sqrt{w_1^2 + w_2^2}, 0, \sqrt{w_3^2 + w_4^2}, 0\right].$$

As $\mathbf{w}$ is a minimizer for I, we have equality in the above inequalities, which leads to

$$|\partial_r w_j|^2 + |\partial_r w_{j+1}|^2 = \left|\partial_r \sqrt{w_j^2 + w_{j+1}^2}\right|^2 \text{ for } j \in \{1, 3\}, \tag{2.14}$$

$$(w_1, w_2) \text{ and } (w_3^2 - w_4^2, 2w_3w_4) \text{ are colinear with a non-negative colinear factor.} \tag{2.15}$$

From (2.14), we deduce that there exist constant unit vectors $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$, $\alpha, \beta \in [0, 2\pi)$, and scalar functions λ and μ such that

$$(w_1, w_2) = \lambda(r)(\cos\alpha, \sin\alpha) \text{ and } (w_3, w_4) = \mu(r)(\cos\beta, \sin\beta).$$

[7] Note that, unlike in (1.7), we are not assuming that the boundary data be uniaxial in this statement.

Case 1: $w_1(R) > 0$ and $w_3(R) > 0$. In this case, we have $\lambda(R) \neq 0$ and $\mu(R) \neq 0$, which implies that $\sin\alpha = \sin\beta = 0$, and hence $w_2 \equiv w_4 \equiv 0$.
Case 2: $w_1(R) > 0$ and $w_3(R) = 0$.[8] This implies that $\lambda(R) \neq 0$ which leads to $\sin\alpha = 0$, and $w_2 \equiv 0$. We need to show that $w_4 \equiv 0$. Since $w_2 = 0$, we have $I[\mathbf{w}] = I[w_0, w_1, 0, w_3, |w_4|]$ and so $(w_0, w_1, 0, w_3, |w_4|)$ is I-minimizing. Thus, we may assume without loss of generality that $w_4 \geqq 0$. As $w_2 \equiv 0$, equation (2.12) reduces to

$$w_4'' + \frac{1}{r}w_4' - \frac{k^2}{4r^2}\,w_4 = w_4\Big(- a^2 + c^2|\mathbf{w}|^2 - \frac{b^2}{\sqrt{6}}w_0 + \frac{b^2}{\sqrt{2}}w_1\Big).$$

By the strong maximum principle we thus have either $w_4 \equiv 0$ or $w_4 > 0$ in $(0, R)$. Assume by contradiction that $w_4 > 0$ in $(0, R)$. As $w_1(R) > 0$, there is some $R' < R$ such that $w_1 > 0$ in (R', R). By (2.15), we have $w_1w_3w_4 \equiv 0$, and so $w_3 \equiv 0$ in (R', R). But this implies that $(w_1, 0)$ is not positively colinear to $(-w_4^2, 0)$ in (R', R) which contradicts (2.15).
Case 3: $w_1(R) = 0$ and $w_3(R) > 0$. This implies that $\mu(R) \neq 0$, $\sin\beta = 0$, and $w_4 \equiv 0$. We need to show that $w_2 \equiv 0$. By (2.15), we have $w_2w_3 \equiv 0$. As $w_4 \equiv 0$, we have $I[\mathbf{w}] = I[w_0, w_1, |w_2|, w_3, 0]$ and so $(w_0, w_1, |w_2|, w_3, 0)$ is I-minimizing. Thus, we may assume without loss of generality that $w_2 \geqq 0$. Also as $w_4 \equiv 0$, equation (2.10) reduces to

$$w_2'' + \frac{1}{r}w_2' - \frac{k^2}{r^2}\,w_2 = w_2\Big(- a^2 + c^2|\mathbf{w}|^2 + \frac{2b^2}{\sqrt{6}}w_0\Big),$$

which implies, in view of the strong maximum principle, that $w_2 \equiv 0$ or $w_2 > 0$ in $(0, R)$. If the latter holds, then as $w_2w_3 \equiv 0$, we would have $w_3 \equiv 0$, which would contradict the fact that $w_3(R) > 0$. We thus have that $w_2 \equiv 0$.
Case 4: $w_1(R) = 0$ and $w_3(R) = 0$. We have

$$\begin{aligned} &2w_0^3 - 6w_0(w_1^2 + w_2^2) + 3w_0(w_3^2 + w_4^2) + 3\sqrt{3}w_1(w_3^2 - w_4^2) + 6\sqrt{3}w_2w_3w_4 \\ &\quad = 2w_0^3 - 6w_0\lambda^2 + 3w_0\mu^2 + 3\sqrt{3}\lambda\mu^2\cos(\alpha - 2\beta) \leqq g(w_0, |\lambda|, \mu) \end{aligned}$$

where

$$g(x, y, z) = 2x^3 - 6xy^2 + 3xz^2 + 3\sqrt{3}yz^2, \qquad (x, y, z) \in \mathbb{R}^3.$$

By Lemma D.1 in Appendix 4, we have $g(x, y, z) \leqq 2(x^2 + y^2 + z^2)^{3/2}$. It thus follows that

$$I[\mathbf{w}] \geqq I[|\mathbf{w}|, 0, 0, 0, 0].$$

Since $\mathbf{w}$ is I-minimizing, we hence have $I[\mathbf{w}] = I[|\mathbf{w}|, 0, 0, 0, 0]$, which implies $w_1 \equiv w_2 \equiv w_3 \equiv w_4 \equiv 0$. $\square$

In Table 2, we summarize the characterization of various symmetries that we introduced for maps in $H^1(B_R, \mathscr{S}_0)$ (in particular, critical points or minimizers of $\mathscr{F}$) in terms of components $w_0, \ldots, w_4$ and $k \neq 0$ even.

[8] For example, (1.7) falls into this case.

Table 2. Characterization of symmetries in the components $w_0, \ldots, w_4$ and $k \in 2\mathbb{Z}\backslash\{0\}$

Symmetries in $H^1(B_R, \mathscr{S}_0)$	Radial components
$SO(2)$-symmetric map	$w_0, \ldots, w_4$
$O(2)$-symmetric map	w_0, w_1, w_3
$\mathbb{Z}_2 \times SO(2)$-symmetric (that is, k-radially symmetric) map	w_0, w_1, w_2
$\mathbb{Z}_2 \times O(2)$-symmetric map	w_0, w_1
$SO(2)$-symmetric minimizer	w_0, w_1, w_3
$\mathbb{Z}_2 \times SO(2)$-symmetric ($k$-radially symmetric) critical point	w_0, w_1

3. Minimizers with k-Fold $O(2)$-Symmetry on Large Disks

In this section, we provide the proof of Theorem 1.5. Instead of working directly with the functional $\mathscr{F}[\cdot; B_R]$ defined in (1.1) we rescale the domain B_R to the unit disk $D \equiv B_1$. We work with a new parameter $\varepsilon = \frac{1}{R}$ and the following rescaled Landau–de Gennes energy functional

$$\mathscr{F}_\varepsilon[Q] := \int_D \left[\frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f_{\text{bulk}}(Q)\right] \mathrm{d}x, \tag{3.1}$$

defined on the set

$$H^1_{Q_b}(D, \mathscr{S}_0) = \left\{Q \in H^1(D, \mathscr{S}_0) : Q = Q_b \text{ on } \partial D\right\}$$

with Q_b given by (1.8). Throughout the section k is an even non-zero integer.

The Euler–Lagrange equation for $\mathscr{F}_\varepsilon$ reads as

$$\varepsilon^2 \Delta Q = -a^2 Q - b^2\left[Q^2 - \frac{1}{3}\mathrm{tr}(Q^2) I_3\right] + c^2 \mathrm{tr}(Q^2) Q \text{ in } D. \tag{3.2}$$

The statement on the uniqueness up to $\mathbb{Z}_2$-conjugation for minimizers of $\mathscr{F}[\cdot; B_R]$ in Theorem 1.5 is equivalent to the following:

Theorem 3.1. *Let $a^2 \geqq 0, b^2, c^2 > 0$ be any fixed constants and $k \in 2\mathbb{Z}\backslash\{0\}$. There exists some $\varepsilon_0 = \varepsilon_0(a^2, b^2, c^2, k) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist exactly two minimizers $Q_\varepsilon^\pm$ of $\mathscr{F}_\varepsilon$ in $H^1_{Q_b}(D, \mathscr{S}_0)$ and these minimizers are k-fold $O(2)$-symmetric but not $\mathbb{Z}_2$-symmetric. Moreover, they are $\mathbb{Z}_2$-conjugate, namely, $Q_\varepsilon^\pm = J Q_\varepsilon^\mp J \neq Q_\varepsilon^\mp$ with J as defined in* (1.18).

3.1. Towards the Proof of Theorem 3.1

To begin with, we note that, in Theorem 3.1, the assertion on the $O(2)$-symmetry of minimizers is a consequence of the uniqueness up to $\mathbb{Z}_2$-conjugation and the fact that the functional $\mathscr{F}_\varepsilon$ and the boundary condition (1.7) are $O(2)$-invariant. We thus focus on discussing the ideas of the proof of the uniqueness up to $\mathbb{Z}_2$-conjugation.

Using standard arguments it is straightforward to show that as $\varepsilon \to 0$ the minimizers of $\mathscr{F}_\varepsilon$ converge, along subsequences, in $H^1_{Q_b}(D, \mathscr{S}_0)$ to the minimizers of the harmonic map problem

$$\mathscr{F}_*[Q] = \int_D \frac{1}{2}|\nabla Q|^2 \,\mathrm{d}x, \quad Q \in H^1_{Q_b}(D, \mathscr{S}_*), \tag{3.3}$$

where

$$H^1_{Q_b}(D,\mathscr{S}_*)=\left\{Q\in H^1(D,\mathscr{S}_*):Q=s_+\left(n\otimes n-\frac{1}{3}I_3\right)\text{ on }\partial D\right\},$$

and $\mathscr{S}_*$ defined in (1.5) is the set of global minimizers of $f_{\text{bulk}}(Q)$, see for example [5,31].

Due to the explicit form of Q_b and the fact that k is even, the minimizers of $\mathscr{F}_*$ in $H^1_{Q_b}(D,\mathscr{S}_*)$ can be written in the form $s_+(n_*\otimes n_*-\frac{1}{3}I_3)$ (see [3]), where n_* minimizes the problem

$$\begin{aligned}&\mathscr{F}_{OF}[v]=\int_D\frac{1}{2}|\nabla v|^2\,\mathrm{d}x,\\&v\in H^1_n(D,\mathbb{S}^2)=\{v\in H^1(D,\mathbb{S}^2):v=n\text{ on }\partial D\}.\end{aligned}\tag{3.4}$$

It is well known, see for example [9, Lemma A.2], that minimizers of (3.4) are conformal and have images in either the upper or lower hemisphere. The compositions of these minimizers with the stereographic projections of the upper and lower hemispheres of $\mathbb{S}^2$ onto the unit disk are the complex maps $z\mapsto z^{k/2}$ or $z\mapsto\bar{z}^{k/2}$, respectively. Therefore $\mathscr{F}_{OF}$ has exactly two minimizers in $H^1_n(D,\mathbb{S}^2)$ which are given by

$$n_*^{\pm}(r\cos\varphi,r\sin\varphi)=\left(\frac{2r^{\frac{k}{2}}\cos\left(\frac{k}{2}\varphi\right)}{1+r^k},\frac{2r^{\frac{k}{2}}\sin\left(\frac{k}{2}\varphi\right)}{1+r^k},\pm\frac{1-r^k}{1+r^k}\right).\tag{3.5}$$

The corresponding minimizers of $\mathscr{F}_*$ are

$$Q_*^{\pm}=s_+\left(n_*^{\pm}\otimes n_*^{\pm}-\frac{1}{3}I_3\right).\tag{3.6}$$

We note that $Q_*^{\pm}$ are smooth and $O(2)$-symmetric but not $\mathbb{Z}_2$-symmetric and we can explicitly write $Q_*^{\pm}$ in terms of the basis tensors $\{E_i\}$ (see (2.2)) as

$$Q_*^{\pm}=w_0^*(r)E_0+w_1^*(r)E_1\pm w_3^*(r)E_3,$$

where

$$w_0^*(r)=s_+\frac{2(1-r^k)^2-4r^k}{\sqrt{6}(1+r^k)^2},\;w_1^*(r)=\frac{4s_+r^k}{\sqrt{2}(1+r^k)^2},\;w_3^*(r)=\frac{4s_+r^{\frac{k}{2}}(1-r^k)}{\sqrt{2}(1+r^k)^2}.$$

It is possible to show that from any sequence of minimizers Q_{ε_k} of $\mathscr{F}_{\varepsilon_k}$ in $H^1_{Q_b}(D,\mathscr{S}_0)$ one can extract a subsequence which converges in $C^{1,\alpha}(\bar{D})$ and $C^j_{loc}(D)$ for any $j\geqq 2$ to either Q_*^+ or Q_*^- (the reasoning requires straightforward modifications of the arguments in [5,32]). Using the energy representation (2.4) one observes that if $Q_\varepsilon=\sum_{i=0}^4 w_{i,\varepsilon}E_i$ is a minimizer of $\mathscr{F}_\varepsilon$ in $H^1_{Q_b}(D,\mathscr{S}_0)$, then the $\mathbb{Z}_2$-conjugate $\tilde{Q}_\varepsilon=JQ_\varepsilon J=\sum_{i=0}^2 w_{i,\varepsilon}E_i-\sum_{i=3}^4 w_{i,\varepsilon}E_i$ of Q_ε is also a minimizer of $\mathscr{F}_\varepsilon$ in $H^1_{Q_b}(D,\mathscr{S}_0)$. Also, if $Q_{\varepsilon'_k}\to Q_*^+$, then $\tilde{Q}_{\varepsilon'_k}\to Q_*^-=JQ_*^+J$ and vice versa. Thus both Q_*^+ and Q_*^- can appear as limits of the sequences of minimizers of $\mathscr{F}_\varepsilon$.

To prove the uniqueness up to $\mathbb{Z}_2$-conjugation of minimizers, one possible approach is to employ the contraction mapping theorem or the implicit function theorem to show that

there are "neighborhoods" $\mathscr{N}^{\pm}$ of $Q_*^{\pm}$ such that when ε is small enough $\mathscr{F}_\varepsilon$ admits at most one critical point in each of $\mathscr{N}^{\pm}$. (†)

In this approach typically the neighborhoods $\mathscr{N}^{\pm}$ are set up in norms relatively stronger than the energy-associated norm. In addition, the norm and thus the neighborhood are dependent on ε. A delicate point is the competition between the size of the neighborhood where one can prove uniqueness and the rate of convergence of minimizing sequences to the limit $Q_*^{\pm}$ (so that one can squeeze all minimizers into the designed neighborhood).

Below we present a roadmap to the proof of Theorem 3.1. Since $Q_*^{\pm}$ are equivalent up to a $\mathbb{Z}_2$-conjugation, it suffices to construct one such neighborhood, say $\mathscr{N}^{+}$ of Q_*^{+}. For simplicity, we will in the sequel drop the superindex $+$, so that $Q_* = Q_*^{+}$, $n_* = n_*^{+}$, etc.

In Subsection 3.2, we will provide a parameterization of suitable neighbourhoods $\mathscr{N}^{\pm}$ where we have a decomposition

$$Q = Q_\sharp + \varepsilon^2 P = s_+\Big(\frac{n_* + \psi}{|n_* + \psi|} \otimes \frac{n_* + \psi}{|n_* + \psi|} - \frac{1}{3} I_3\Big) + \varepsilon^2 P$$

with $\varepsilon^2 P$ being a "transversal component" of Q and $Q_\sharp$ being a (non-orthogonal) "projection" onto the limit manifold $\mathscr{S}_*$.

In Subsection 3.3 we employ the above parameterization to obtain a new representation of the Euler–Lagrange equations (3.2) in terms of the variables ψ and P. In particular, we will derive a coupled system of equations for ψ and P with the following properties:

1. One equation is of the form

$$L_\parallel \psi = \text{Lagrange multiplier terms} + F[\varepsilon, \psi, P], \tag{3.7}$$

   where the operator $L_\parallel = -\Delta - |\nabla n_*|^2$ is the linearized harmonic map operator at the minimizer n_* of the problem (3.4). See (3.15) for the exact equation.
2. The other equation is of the form

$$L_{\varepsilon,\perp} P = \text{Lagrange multiplier terms} + s_+ \Delta(n_* \otimes n_*) + G[\varepsilon, \psi, P], \tag{3.8}$$

   with the linear operator $L_{\varepsilon,\perp} P = -\varepsilon^2 \Delta P + b^2 s_+ P + 2(c^2 s_+ - b^2)(P n_* \cdot n_*) Q_*$. See (3.16) for the exact form.

We will see that, although the nonlinear operators F and G are second order in the fields ψ and P, they are 'super-linear' and are 'small' when ψ and P are suitably 'small'. Subsection 3.4 is devoted to study the operator $L_\parallel$ and in Subsection 3.5 we concentrate on the operator $L_{\varepsilon,\perp}$.

In Subsection 3.6, using previously derived properties of $L_{\parallel}$, we revisit (3.7) and study the dependence of its solution ψ_ε (with zero Dirichlet boundary condition) as a map of P. In order to balance the rate of convergence of the sequences of minimizers and the size of the neighborhoods $\mathscr{N}^{\pm}$ it will be convenient to measure the size of P with respect to an ε-dependent H^2 norm, specifically defined as

$$\|P\|_\varepsilon := \|P\|_{L^2(D)} + \varepsilon\|\nabla P\|_{L^2(D)} + \varepsilon^2\|\nabla^2 P\|_{L^2(D)}. \tag{3.9}$$

We first show that, for $P \in H_0^1 \cap H^2(D, \mathscr{S}_0)$ with $\|P\|_{L^2(D)} = O(1)$ and $\|\nabla^2 P\|_{L^2(D)} = o(\varepsilon^{-2})$, one can solve (3.7) for $\psi_\varepsilon = \psi_\varepsilon(P) \in H_0^1 \cap H^2(D, \mathbb{R}^3)$. Furthermore, when P is measured with respect to the norm $\|\cdot\|_\varepsilon$ above, ψ_ε is Lipschitz with respect to P with Lipschitz constant $O(\varepsilon)$ (see Proposition 3.12). Using the Lipschitz estimate above, we show that the map $P \mapsto L_{\varepsilon,\perp}^{-1} G[\varepsilon, \psi_\varepsilon(P), P]$ is contractive. This proves the uniqueness statement formulated informally above in relation (†). See Proposition 3.14 in Subsection 3.7. In Subsection 3.8, using convergence results from [32] and the results presented above we prove Theorem 3.1. Finally, the proof of Theorem 1.5 is done in Subsection 3.9.

3.2. A Parametrization in Small H^2-Neighborhoods of Q_*

In this section we show that every $Q \in H^1_{Q_b}(D, \mathscr{S}_0) \cap H^2(D, \mathscr{S}_0)$ sufficiently close to a minimizer Q_* of $\mathscr{F}_*$ can be decomposed in a special way (see (3.10)) that takes into account the geometry of the limit manifold $\mathscr{S}_*$ and the way it embeds into the space $\mathscr{S}_0$ of Q-tensors.

Since $\mathscr{S}_*$ is a smooth compact submanifold of $\mathscr{S}_0$, we can find a neighborhood $N(\mathscr{S}_*)$ of $\mathscr{S}_*$ in $\mathscr{S}_0$, such that for every $B \in N(\mathscr{S}_*)$, there exists a unique $B_{ort} \in \mathscr{S}_*$ such that $|B - B_{ort}| = \mathrm{dist}(B, \mathscr{S}_*)$ where $|\cdot|$ stands for the norm associated to the Frobenius scalar product. Furthermore the projection $B \mapsto B_{ort}$ is a smooth map from $N(\mathscr{S}_*)$ onto $\mathscr{S}_*$. See Fig. 1.

Although the above orthogonal projection suffices for many purposes, it is somewhat more convenient in our current setting to work with a different projection which is more adapted to Q_*. Let $n_* = n_*^+$ be as in (3.5) and $Q_* = Q_*^+ = s_+(n_* \otimes n_* - \frac{1}{3}I_3)$. We denote by $T_{Q_*}\mathscr{S}_* = T_{Q_*(x)}\mathscr{S}_*$ the tangent space to the *limit manifold* $\mathscr{S}_*$ at $Q_*(x)$ and by $(T_{Q_*}\mathscr{S}_*)^\perp$ its orthogonal complement in $\mathscr{S}_0 \approx \mathbb{R}^5$, which is normal to $\mathscr{S}_*$ at $Q_*(x)$. It is known that $(T_{Q_*}\mathscr{S}_*)^\perp$ consists of all matrices in $\mathscr{S}_0$ commuting with Q_*; see [32, Eq. (3.2)]. In particular, all matrices in $(T_{Q_*}\mathscr{S}_*)^\perp$ admit n_* as an eigenvector.[9]

We want to show that every Q in a "sufficiently small neighborhood" of Q_* decomposes as

$$Q(x) := \underbrace{s_+\left(v(x) \otimes v(x) - \frac{1}{3}I_3\right)}_{\text{belongs to } \mathscr{S}_*} + \text{part transversal to } \mathscr{S}_* \tag{3.10}$$

[9] Recall that the eigenspace of Q_* corresponding to the eigenvalue $\lambda_* = \frac{2}{3}s_+$ is of dimension one and generated by n_*; therefore, if $AQ_* = Q_*A$, then An_* belongs to this eigenspace, that is, An_* is parallel to n_*.

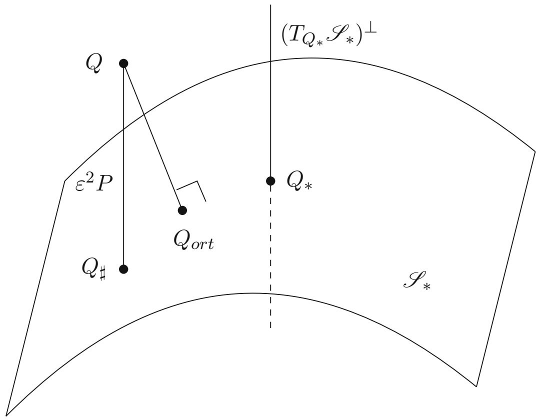

Fig. 1. The decomposition in Lemma 3.2 vs orthogonal projection

so that $Q(x) - s_+\left(v(x)\otimes v(x) - \frac{1}{3}I_3\right) \in (T_{Q_*}\mathscr{S}_*)^\perp$, which will be useful later. See Fig. 1. We specify the result in the following lemma:

Lemma 3.2. *Let $n_* = n_*^+$ and $Q_* = Q_*^+$ be as in (3.5) and (3.6). There exist $\gamma > 0$ and some large $C_0 > 0$ such that for every $\varepsilon > 0$ and every $Q \in H^1_{Q_b}(D, \mathscr{S}_0) \cap H^2(D, \mathscr{S}_0)$ with $\|Q - Q_*\|_{H^2(D)} \leqq \frac{1}{C_0}$ we can uniquely write*

$$Q = Q_\sharp + \varepsilon^2 P, \tag{3.11}$$

where $Q_\sharp$ and P satisfy that

- $Q_\sharp \in H^1_{Q_b}(D, \mathscr{S}_*) \cap H^2(D, \mathscr{S}_*)$,
- $P \in H^1_0(D, \mathscr{S}_0) \cap H^2(D, \mathscr{S}_0)$ *with* $P(x) \in (T_{Q_*}\mathscr{S}_*)^\perp$ *for* $x \in D$,
- $\|Q_\sharp - Q_*\|_{H^2(D)} \leqq \gamma \|Q - Q_*\|_{H^2(D)}$,
- *and* $\varepsilon^2\|P\|_{H^2(D)} \leqq \gamma\,\|Q - Q_*\|_{H^2(D)}$.

Furthermore, there exists a unique $\psi \in H^1_0(D, \mathbb{R}^3) \cap H^2(D, \mathbb{R}^3)$ with $\psi \cdot n_* = 0$ almost everywhere in D such that $\|\psi\|_{H^2(D)} \leqq \gamma\|Q - Q_*\|_{H^2(D)}$ and

$$Q_\sharp = s_+\Big(\frac{n_* + \psi}{|n_* + \psi|} \otimes \frac{n_* + \psi}{|n_* + \psi|} - \frac{1}{3}I_3\Big). \tag{3.12}$$

Remark 3.3. In the above lemma, we have deliberately written the "transversal" component of Q as $\varepsilon^2 P$ even though ε plays no role at the moment. In [32], it is shown that, in a similar setting, if Q_ε is a minimizer for $\mathscr{F}_\varepsilon$, then its "transversal" contribution is of size ε^2 in some appropriate topology. In the setting of the present paper, we will show this holds in the $L^2(D, \mathscr{S}_0)$-topology; see (3.44) below. This rate of convergence however does not hold in the $H^2(D, \mathscr{S}_0)$-topology.[10] This is

[10] For such a rate of convergence would imply in view of Proposition 3.12 below that $\|Q_\varepsilon - Q_*\|_{H^2(D,\mathscr{S}_0)} = O(\varepsilon^2)$, which would further imply that the limit of $\varepsilon^{-2}(Q_\varepsilon - Q_*)$ has zero trace on ∂D, which would contradict [32, Theorem 2, Eq. (2.14)]. See also [5] for a similar statement in the Ginzburg–Landau setting.

related to the comment we made earlier on the fact that the sets $\mathscr{N}^{\pm}$ in (†) are ε-dependent.

Remark 3.4. It should be noted that the map ψ appearing in the representation of $Q_\sharp$ belongs to a linear space (as ψ is orthogonal to n_*) as opposed to $Q_\sharp$ that belongs to a nonlinear set (as its values are being constrained in $\mathscr{S}_*$).

Proof. Since $\mathscr{S}_*$ is a smooth submanifold of $\mathscr{S}_0$, there exists for every point $B_* \in \mathscr{S}_*$ a neighborhood U_{B_*} of B_* in $\mathscr{S}_0$ such that $\mathscr{S}_* \cap U_{B_*}$ is a graph over the tangent plane $T_{B_*}\mathscr{S}_*$. We then select local Cartesian-type coordinates $\{x_1, \ldots, x_5\}$ of $\mathscr{S}_0 \approx \mathbb{R}^5$ such that B_* corresponds to the origin, $T_{B_*}\mathscr{S}_*$ coincides with $\{(x_1, x_2, 0, 0, 0) : x_1, x_2 \in \mathbb{R}\}$ and $\mathscr{S}_* \cap U_{B_*}$ is given by $\{(x_1, x_2, u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)) : (x_1, x_2) \in \widetilde{\mathcal{U}}\}$ for some open set $\widetilde{\mathcal{U}} \subset \mathbb{R}^2$ and some smooth function $u = (u_1, u_2, u_3) : \widetilde{\mathcal{U}} \to \mathbb{R}^3$ with $u(0) = 0$ and $\nabla u(0) = 0$. Define a projection $\mathscr{P}_{B_*}$ from U_{B_*} to $\mathscr{S}_*$ by

$$\mathscr{P}_{B_*}(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, u(x_1, x_2)).$$

One can check that $\mathscr{P}_{B_*}(B)$ is well-defined (that is independent of local charts) and smooth as a function of two variables $B \in \mathscr{S}_0$ and $B_* \in \mathscr{S}_*$. Furthermore, $\mathscr{P}_{B_*}$ is the unique projection with the property $B - \mathscr{P}_{B_*}(B) \in (T_{B_*}\mathscr{S}_*)^{\perp}$.

As D is two dimensional, maps in $H^2(D, \mathscr{S}_0)$ are continuous. Thus, there exists some large constant $C_0 > 0$ such that whenever

$$\|Q - Q_*\|_{H^2(D)} \leqq \frac{1}{C_0}, \tag{3.13}$$

there holds $Q(x) \in U_{Q_*(x)}$ for all $x \in D$. The decomposition (3.11) is achieved by

$$Q_\sharp(x) = \mathscr{P}_{Q_*(x)}(Q(x)) \text{ and } P(x) = \varepsilon^{-2}(Q(x) - Q_\sharp(x)) \text{ for } x \in D.$$

We now proceed to check the desired properties of $Q_\sharp$ and P. First, we have

$$Q_\sharp(x) - Q_*(x) = \mathscr{P}_{Q_*(x)}(Q(x)) - \mathscr{P}_{Q_*(x)}(Q_*(x)).$$

Using the smoothness of $\mathscr{P}$ in both variables, we obtain the claimed control of $\|Q_\sharp - Q_*\|_{H^2(D)}$ in terms of $\|Q - Q_*\|_{H^2(D)}$. Furthermore we have $Q - Q_\sharp = Q - Q_* + Q_* - Q_\sharp$ which also provides the control of the H^2-norm of $\varepsilon^2 P = Q - Q_\sharp$ in terms of the H^2-norm of $Q - Q_*$, as claimed.

We turn to the second part of the lemma. Note that $Q_\sharp \in H^2(D, \mathscr{S}_*)$ is continuous. As D is simply connected and $\mathscr{S}_*$ can be topologically identified with a projective plane, a standard result in topology about covering spaces implies that there is a unique continuous function $v \in C^0(D, \mathbb{S}^2)$ such that

$$Q_\sharp = s_+ \left(v \otimes v - \frac{1}{3} I_3 \right) \text{ in } D \text{ and } v = n \text{ on } \partial D.$$

Furthermore, by [3, Theorem 2], we have $v \in W^{1,p}(D, \mathbb{S}^2)$ for any $p \geqq 2$.

Note that $\nabla_k(Q_\sharp)_{ij} = s_+(\nabla_k v_i\, v_j + \nabla_k v_j\, v_i)$, and so, as $|v| = 1$,

$$\nabla_k v_i = \frac{1}{s_+}\nabla_k(Q_\sharp)_{ij}\, v_j,$$

from which one can easily deduce that $v \in H^2(D, \mathbb{S}^2)$.

Next note that, as $Q_\sharp - Q_* = s_+(v \otimes v - n_* \otimes n_*)$, we have

$$\begin{aligned} &|Q_\sharp - Q_*|^2 = 2s_+^2(1 - (v \cdot n_*)^2) \quad \text{and} \\ &(Q_\sharp - Q_*)n_* = s_+[(v \cdot n_*)v - n_*]. \end{aligned} \tag{3.14}$$

Taking C_0 large enough in (3.13) and using equality (3.14), we obtain $(v \cdot n_*)^2 \geq \frac{1}{4}$. Since both v and n_* are continuous and coincide at the boundary we deduce $v \cdot n_* \geqq \frac{1}{2}$. Therefore we can define

$$\psi = \frac{1}{v \cdot n_*}\, v - n_*.$$

Observe that the above is equivalent to $\left(\psi \cdot n_* = 0 \text{ and } v = \frac{n_*+\psi}{|n_*+\psi|}\right)$, which gives the uniqueness of ψ. On the other hand, one has

$$\psi = \frac{(v \cdot n_*)v - n_*}{(v \cdot n_*)^2} + \frac{n_*(1 - (v \cdot n_*)^2)}{(v \cdot n_*)^2}.$$

Recalling relations (3.14) we can represent $\psi = G(n_*, Q_\sharp - Q_*)$, where G is a smooth map provided that $|Q_\sharp - Q_*|$ is small. We hence obtain $\|\psi\|_{H^2(D)} \leqq C\|Q_\sharp - Q_*\|_{H^2(D)} \leqq \gamma\|Q - Q_*\|_{H^2(D)}$. This concludes the proof. $\square$

3.3. The Euler–Lagrange Equations

In this subsection we rewrite the Euler–Lagrange equations (3.2) for $\mathscr{F}_\varepsilon$ in terms of the variables ψ and P introduced in Lemma 3.2. This new form of the Euler–Lagrange equations will be used in the subsequent analysis.

Lemma 3.5. *Let $Q \in H^1_{Q_b}(D, \mathscr{S}_0) \cap H^2(D, \mathscr{S}_0)$ be a critical point of $\mathscr{F}_\varepsilon$ for some $\varepsilon > 0$, and n_* and Q_* be given by (3.5) and (3.6). Suppose that $\|Q - Q_*\|_{H^2(D)}$ is sufficiently small and let $Q_\sharp$, P and ψ be as in Lemma 3.2. Then ψ and P satisfy the following equations*

$$-\Delta\psi - |\nabla n_*|^2\,\psi = \lambda_\varepsilon\, n_* + A[\psi] + \varepsilon^2\, B_\varepsilon[\psi, P], \tag{3.15}$$

$$\begin{aligned} &-\varepsilon^2\Delta P + b^2\, s_+\, P + 2(c^2\, s_+ - b^2)\,(P n_* \cdot n_*)Q_* \\ &\quad = F_\varepsilon + s_+\Delta(n_* \otimes n_*) + C_\varepsilon[\psi, P] - \frac{1}{3}\mathrm{tr}(C_\varepsilon[\psi, P])I_3, \end{aligned} \tag{3.16}$$

$$\psi \cdot n_* = 0, \quad P \in (T_{Q_*}\mathscr{S}_*)^\perp \quad \textit{in } D, \tag{3.17}$$

where $\lambda_\varepsilon(x)$ is a Lagrange multiplier accounting for the constraint $\psi \cdot n_* = 0$, $F_\varepsilon(x) \in T_{Q_*}\mathscr{S}_*$ is a Lagrange multiplier accounting for the constraint $P \in (T_{Q_*}\mathscr{S}_*)^\perp$, and maps A, B_ε, C_ε are defined in equations (B.9), (B.10) and (B.11) in Appendix 4.

In Lemma 3.5 above, we do not provide the exact forms of A, B_ε, C_ε nor indicate their explicit dependence on x as we show later (see the proof of Lemma 3.5) that these are lower order terms that do not play a role in our analysis. We will only use their properties summarized in the following proposition.

Proposition 3.6. *Let $\varepsilon \in (0,1)$, n_* and Q_* be given by (3.5) and (3.6), and let A, B_ε and C_ε be the operators appearing in Lemma 3.5, defined in (B.9), (B.10), (B.11) in Appendix 4. Then, for $\psi \in H_0^1(D,\mathbb{R}^3) \cap H^2(D,\mathbb{R}^3)$, $P \in H_0^1(D,\mathscr{S}_0) \cap H^2(D,\mathscr{S}_0)$ satisfying $\psi \cdot n_* = 0$ and $P \in (T_{Q_*}\mathscr{S}_*)^\perp$ in D, we have the following:*

$$A[0] = 0, \tag{3.18}$$

$$\begin{aligned}\|A[\psi] - A[\tilde{\psi}]\|_{L^2(D)} &\leqq C(\|\psi\|_{H^2(D)} + \|\tilde{\psi}\|_{H^2(D)})\\ &\quad\times (1 + \|\psi\|_{H^2(D)} + \|\tilde{\psi}\|_{H^2(D)})\|\psi - \tilde{\psi}\|_{H^2(D)},\end{aligned} \tag{3.19}$$

$$\|B_\varepsilon[0,P]\|_{L^2(D)} \leqq C\|P\|_{H^1(D)}, \tag{3.20}$$

$$\begin{aligned}&\|B_\varepsilon[\psi,P] - B_\varepsilon[\tilde{\psi},P]\|_{L^2(D)}\\ &\quad\leqq C\Big[\|\nabla^2 P\|_{L^2(D)} + (1+\varepsilon^2\|P\|_{H^2(D)})\|P\|^2_{L^4(D)}\Big]\|\psi - \tilde{\psi}\|_{H^2(D)},\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\|B_\varepsilon[\psi,P] - B_\varepsilon[\psi,\tilde{P}]\|_{L^2(D)}\\ &\quad\leqq C\|P - \tilde{P}\|_{H^1(D)} + C\|\psi\|_{H^2(D)}\\ &\quad\Big(\|P - \tilde{P}\|_{H^2(D)} + (\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)})(1 + \varepsilon^2\|P\|_{H^2(D)}\\ &\quad + \varepsilon^2\|\tilde{P}\|_{H^2(D)})\|P - \tilde{P}\|_{L^2(D)}\Big),\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\|C_\varepsilon(\psi,P) - C_\varepsilon(\tilde{\psi},\tilde{P})\|_{L^2(D)}\\ &\quad\leqq C(1 + \|\psi\|_{H^2(D)} + \|\tilde{\psi}\|_{H^2(D)})^2\|\psi - \tilde{\psi}\|_{H^2(D)}\\ &\quad + C(\|P\|_{L^2(D)} + \|\tilde{P}\|_{L^2(D)})\\ &\quad (1 + \varepsilon^2(\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)}))\|\psi - \tilde{\psi}\|_{H^2(D)}\\ &\quad + C(\|\psi\|_{H^2(D)} + \|\tilde{\psi}\|_{H^2(D)})\|P - \tilde{P}\|_{L^2(D)}\\ &\quad + C\varepsilon^2(\|P\|_{L^4(D)} + \|\tilde{P}\|_{L^4(D)})\\ &\quad (1 + \varepsilon^2(\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)}))\|P - \tilde{P}\|_{H^1(D)},\end{aligned} \tag{3.23}$$

with C denoting various constants independent of ε and the functions appearing in the inequalities.

The proofs of Lemma 3.5 and Proposition 3.6 are lengthy though elementary. We postpone them to Appendix 4.

3.4. The Linearized Harmonic Map Problem

In this subsection we briefly study the properties of the operator $L_\parallel = -\Delta - |\nabla n_*|^2$ appearing on the left hand side of (3.15), that is the linearized harmonic map operator at n_* given by (3.5), as well as its inverse $L_\parallel^{-1}$.

Proposition 3.7. *For every $f \in L^2(D, \mathbb{R}^3)$, the minimization problem*

$$\min\left\{ \int_D [|\nabla\zeta|^2 - |\nabla n_*|^2\,\zeta^2 - f\cdot\zeta]\,\mathrm{d}x : \zeta \in H_0^1(D,\mathbb{R}^3),\ \zeta\cdot n_* = 0 \text{ almost everywhere in } D\right\}$$

admits a minimizer which is the unique solution to the problem

$$\begin{cases} L_\parallel\zeta \equiv -\Delta\zeta - |\nabla n_*|^2\zeta = f + \lambda(x)\,n_* & \text{in } D,\\ \zeta\cdot n_* = 0 & \text{in } D,\\ \zeta = 0 & \text{on } \partial D,\end{cases} \tag{3.24}$$

where λ is a Lagrange multiplier.[11]

Using Proposition 3.7 we can define the inverse operator $L_\parallel^{-1}$.

Definition 3.8. For $f \in L^2(D,\mathbb{R}^3)$, we define $L_\parallel^{-1} f \in H_0^1(D,\mathbb{R}^3)$ to be the unique solution to (3.24).

The proof of Proposition 3.7 is a standard argument using Lax-Milgram's theorem and the strict stability of n_*. For completeness, we give the proof in Appendix 4.

In the following lemma we prove some useful properties of $L_\parallel^{-1}$ required for our analysis.

Lemma 3.9. *The range of the operator $L_\parallel^{-1}$ over L^2-data is*

$$X := \{\psi \in H_0^1(D,\mathbb{R}^3)\cap H^2(D,\mathbb{R}^3) : \psi\cdot n_* = 0 \text{ in } D\}. \tag{3.25}$$

Furthermore, there exists some positive constant C such that, for $f \in L^2(D,\mathbb{R}^3)$,

$$\|L_\parallel^{-1} f\|_{H^1(D)} \leqq C\|f\|_{H^{-1}(D)} \text{ and } \|L_\parallel^{-1} f\|_{H^2(D)} \leqq C\|f\|_{L^2(D)}.$$

Proof. Let $f \in L^2(D,\mathbb{R}^3)$ and $\zeta \in H_0^1(D,\mathbb{R}^3)$ be the solution of (3.24). We first show that $\zeta \in H^2(D,\mathbb{R}^3)$. Let us fix some $\xi \in C_c^\infty(D)$. Testing (3.24) against $\xi\, n_*$ and noting that $n_*\cdot\zeta = 0 = \Delta n_*\cdot\zeta$, we obtain by integration by parts

$$\begin{aligned}\int_D \xi(f\cdot n_* + \lambda)\,\mathrm{d}x &= \int_D \nabla\zeta\cdot\nabla(\xi n_*)\,\mathrm{d}x = -\int_D \zeta\cdot\Delta(\xi n_*)\,\mathrm{d}x\\ &= -2\int_D \zeta\cdot(\nabla n_*(\nabla\xi))\,\mathrm{d}x = 2\int_D \xi\nabla\zeta\cdot\nabla n_*\,\mathrm{d}x.\end{aligned}$$

Since this is true for all $\xi \in C_c^\infty(D)$, it follows that

$$\lambda = 2\nabla\zeta\cdot\nabla n_* - f\cdot n_* \in L^2(D). \tag{3.26}$$

By elliptic regularity for (3.24), we conclude that $\zeta \in H^2(D,\mathbb{R}^3)$.

[11] The expression of the Lagrange multiplier λ is given in (3.26).

We next turn to estimating ζ. Testing (3.24) against ζ and using Lemma C.2, we obtain

$$\begin{aligned} c_0\|\zeta\|^2_{H^1(D)} &\leqq \int_D [|\nabla\zeta|^2 - |\nabla n_*|^2\,|\zeta|^2]\,\mathrm{d}x \\ &= \int_D f\cdot\zeta\,\mathrm{d}x \leqq \|f\|_{H^{-1}(D)}\,\|\zeta\|_{H^1(D)}. \end{aligned}$$

This implies $\|L_{\|}^{-1} f\|_{H^1(D)} = \|\zeta\|_{H^1(D)} \leqq C\,\|f\|_{H^{-1}(D)}$. Using this estimate in (3.26) we have $\|\lambda\|_{H^{-1}(D)} \leqq C\,\|f\|_{H^{-1}(D)}$ and $\|\lambda\|_{L^2(D)} \leqq C\|f\|_{L^2(D)}$. Employing elliptic estimates for (3.24) we obtain $\|\zeta\|_{H^2(D)} \leqq C\|f\|_{L^2(D)}$. □

3.5. The Transversal Linearized Problem

In this section we study the linear operator appearing on the left hand side of (3.16)

$$L_{\varepsilon,\perp}P = -\varepsilon^2\Delta P + b^2\,s_+\,P + 2(c^2\,s_+ - b^2)\,(Pn_*\cdot n_*)Q_*. \tag{3.27}$$

As in the previous subsection we would like to define the inverse operator $L^{-1}_{\varepsilon,\perp}$ and prove some properties required for our analysis.

We claim that $P\mapsto b^2\,s_+\,P+2(c^2\,s_+-b^2)\,(Pn_*\cdot n_*)Q_*$ is a monotone linear operator, namely

$$\begin{aligned} &b^2\,s_+\,|P|^2 + 2s_+(c^2\,s_+ - b^2)\,(Pn_*\cdot n_*)^2 \\ &\geqq \min\left(2a^2 + \frac{b^2}{3}s_+, b^2 s_+\right)|P|^2, \forall P(x)\in(T_{Q_*}\mathscr{S}_*)^\perp. \end{aligned}\tag{3.28}$$

Indeed, recall that n_* is an eigenvector of $P\in(T_{Q_*}\mathscr{S}_*)^\perp$. Thus, in some orthonormal basis of $\mathbb{R}^3$, P takes the form $\operatorname{diag}(\lambda_1,\lambda_2,-\lambda_1-\lambda_2)$ with $Pn_* = \lambda_1 n_*$. It is not hard to see that this implies $(Pn_*\cdot n_*)^2 = \lambda_1^2 \leqq \frac{4}{3}(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) = \frac{2}{3}|P|^2$. The inequality (3.28) follows in view of the identity $-a^2 - \frac{b^2}{3}s_+ + \frac{2c^2}{3}s_+^2 = 0$.[12]

Using (3.28) and Lax-Milgram's theorem in the Hilbert space

$$\{P\in H^1_0(D,\mathscr{S}_0) : P\in(T_{Q_*}\mathscr{S}_*)^\perp \text{ almost everywhere in } D\},$$

one can easily show that, for every $q\in L^2(D,\mathscr{S}_0)$, there exists a unique solution $P\in H^1_0(D,\mathscr{S}_0)$ to the problem

$$\begin{cases} L_{\varepsilon,\perp}P = q + F(x) & \text{in } D,\\ \quad\;\; P\in(T_{Q_*}\mathscr{S}_*)^\perp & \text{almost everywhere in } D,\\ \quad\;\; P = 0 & \text{on } \partial D,\end{cases}\tag{3.29}$$

[12] Alternatively, one can argue that the operator $P\mapsto b^2\,s_+\,P+2(c^2\,s_+-b^2)\,(Pn_*\cdot n_*)Q_*$ is an automorphism of $(T_{Q_*}\mathscr{S}_*)^\perp$ which has eigenvalues $2a^2+\frac{b^2}{3}s_+$ along the direction parallel to Q_* and $b^2 s_+$ along the directions perpendicular to Q_*, which also gives (3.28).

where $F(x) \in T_{Q_*}\mathscr{S}_*$ is a Lagrange multiplier accounting for the constraint $P \in (T_{Q_*}\mathscr{S}_*)^\perp$ almost everywhere in D. The precise expression for F will be given later, see (3.32), and we will prove that $F \in L^2(D)$.

Therefore we have the following definition:

Definition 3.10. For $q \in L^2(D, \mathscr{S}_0)$, we define $L_{\varepsilon,\perp}^{-1} q \in H_0^1(D, \mathscr{S}_0)$ to be the unique solution to (3.29).

We summarise properties of the operator $L_{\varepsilon,\perp}^{-1}$ in the following lemma:

Lemma 3.11. *For every $\varepsilon > 0$, the range of the operator $L_{\varepsilon,\perp}^{-1} : L^2(D, \mathscr{S}_0) \to H_0^1(D, \mathscr{S}_0)$ is*

$$Y := \{P \in H_0^1(D, \mathscr{S}_0) \cap H^2(D, \mathscr{S}_0) : P \in (T_{Q_*}\mathscr{S}_*)^\perp \text{ in } D\}. \tag{3.30}$$

Furthermore, there exists $C > 0$ such that, for every $0 < \varepsilon < 1$,

$$\|L_{\varepsilon,\perp}^{-1} q\|_\varepsilon \leqq C\|q\|_{L^2(D)} \text{ for all } q \in L^2(D, \mathscr{S}_0),$$

where $\|\cdot\|_\varepsilon$ was defined in (3.9).

Proof. In the proof C will denote some generic constant which varies from line to line but is always independent of ε.

Let us fix some $q \in L^2(D, \mathscr{S}_0)$ and let P be the solution of (3.29). Testing (3.29) against P and using (3.28), we obtain

$$\|P\|_{L^2(D)} + \varepsilon\|\nabla P\|_{L^2(D)} \leqq C\|q\|_{L^2(D)}. \tag{3.31}$$

Next, we would like to show that $P \in H^2(D, \mathscr{S}_0)$. Let $\Pi = \Pi_x$ be the orthogonal projection of $\mathscr{S}_0$ onto $T_{Q_*(x)}\mathscr{S}_*$. Then, the first equation of (3.29) is equivalent to

$$F(x) = \Pi_x(L_{\varepsilon,\perp}P(x) - q(x)). \tag{3.32}$$

Here, we naturally extended Π to distributions, in particular to $\Delta P \in H^{-1}$, by defining $\langle \Pi(\Delta P), \zeta\rangle := \langle \Delta P, \Pi(\zeta)\rangle$ for every test function $\zeta \in C_c^\infty(D, \mathscr{S}_0)$. Therefore, to show that $F \in L^2(D, \mathscr{S}_0)$, it is enough to show that

$$\begin{gathered}\Pi(\Delta P) \in L^2(D, \mathscr{S}_0) \text{ for any } P \in H_0^1(D, \mathscr{S}_0), \text{ satisfying} \\ P(x) \in (T_{Q_*}\mathscr{S}_*)^\perp \text{ almost everywhere in } D.\end{gathered} \tag{3.33}$$

In fact, we show

$$\|\Pi(\Delta P)\|_{L^2(D)} \leqq C\|P\|_{H^1(D)} \tag{3.34}$$

for all $P \in C_c^\infty(D, \mathscr{S}_0)$ satisfying $P(x) \in (T_{Q_*}\mathscr{S}_*)^\perp$ in D. (A straightforward density argument using the fact that Q_* is smooth then yields (3.33).) To this end we use the following formula for Π which was computed in [32, Eq. (3.4)]:[13]

$$\Pi(A) = A + \frac{2}{s_+^2}\left(\frac{1}{3}s_+A - AQ_* - Q_*A\right)\left(Q_* - \frac{1}{6}s_+ I_3\right)$$

[13] The brackets below indeed commute thanks to $Q_*^2 - \frac{1}{3}s_+Q_* - \frac{2}{9}s_+^2 I_3 = 0$.

$$= A + \frac{2}{s_+^2}\left(Q_* - \frac{1}{6}s_+ I_3\right)\left(\frac{1}{3}s_+ A - AQ_* - Q_* A\right), \qquad \text{for all } A \in \mathscr{S}_0.$$

Since $P(x) \in (T_{Q_*}\mathscr{S}_*)^\perp$ in D, $\Pi(P) = 0$ in D and so $\Delta\Pi(P) = 0$ in D. On the other hand, as Q_* is smooth, it follows from the above formula for Π, applied to P and ΔP, that

$$|\Delta\Pi(P) - \Pi(\Delta P)| \leqq C(|\nabla P| + |P|).$$

Combining the above two facts, we obtain (3.34) and hence (3.33).

It follows that $F \in L^2(D, \mathscr{S}_0)$ and $P \in H^2(D, \mathscr{S}_0)$. Moreover, for $0 < \varepsilon < 1$,

$$\|F\|_{L^2(D)} \leqq C(\varepsilon^2\|\nabla P\|_{L^2(D)} + \|P\|_{L^2(D)} + \|q\|_{L^2(D)}). \tag{3.35}$$

Using previously established estimates (3.31) and (3.35) we obtain that $\|F\|_{L^2(D)} \leqq C\|q\|_{L^2(D)}$. Returning to the first equation in (3.29), elliptic estimates yield $\varepsilon^2\|\nabla^2 P\|_{L^2(D)} \leqq C\,\|q\|_{L^2(D)}$. □

3.6. Solution of (3.15) for Given P

In this section we solve equation (3.15) for given P. The properties of the map $P \mapsto \psi_\varepsilon(P)$ obtained in this section will be used later in proving uniqueness of the critical point of $\mathscr{F}_\varepsilon$ in a small neighbourhood of Q_* using fixed point arguments.

We define the following set

$$\mathcal{U}_{\varepsilon,C_1,C_2} := \left\{P \in Y : \varepsilon^2\|\nabla^2 P\|_{L^2(D)} \leqq \frac{1}{C_2}, \|P\|_{L^2(D)} \leqq C_1\right\}, \tag{3.36}$$

where Y is given in (3.30). Note that, by integration by parts,

$$\varepsilon\|\nabla P\|_{L^2(D)} \leqq C_1^{1/2}\, C_2^{-\frac{1}{2}} \text{ for all } P \in \mathcal{U}_{\varepsilon,C_1,C_2}. \tag{3.37}$$

Proposition 3.12. *Let X be defined by* (3.25)*. For every $C_1 > 0$, there exist large $C_2 > 1$ and small $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$,*

(i) *For every $P \in \mathcal{U}_{\varepsilon,C_1,C_2}$, there exists a unique $\psi_\varepsilon(P) \in X$ satisfying simultaneously equation (3.15) and $\|\psi_\varepsilon(P)\|_{H^2(D)} \leqq \frac{1}{C_2}$.*
Furthermore, there exists $C_3 > 0$ (depending on C_1, C_2) such that:

(ii) *For every $P \in \mathcal{U}_{\varepsilon,C_1,C_2}$, $\|\psi_\varepsilon(P)\|_{H^2(D)} \leqq C_3\varepsilon^2\|P\|_{H^1(D)}$. In particular, $\psi_\varepsilon(0) = 0$.*

(iii) *For every $P, \tilde{P} \in \mathcal{U}_{\varepsilon,C_1,C_2}$,*

$$\|\psi_\varepsilon(P) - \psi_\varepsilon(\tilde{P})\|_{H^2(D)} \leqq C_3\varepsilon\|P - \tilde{P}\|_\varepsilon,$$

where $\|\cdot\|_\varepsilon$ is defined in (3.9)*.*

Proof. Let us fix some $C_1 > 0$. In this proof C will denote some generic constant which may depend on C_1, a^2, b^2, c^2 but is independent of ε (and C_2 and P which will appear below).

For $P \in Y$, define an operator $K_{\varepsilon,P} : X \to X$ by

$$K_{\varepsilon,P}(\psi) = L_{\parallel}^{-1}(A[\psi] + \varepsilon^2 B_\varepsilon[\psi, P]),$$

where $L_{\parallel}^{-1}$ is given in Definition 3.8, and A and B_ε are the operators appearing on the right hand side of (3.15).

Proof of (i): It suffices to show that, for sufficiently large C_2 and all $P \in \mathcal{U} := \mathcal{U}_{\varepsilon,C_1,C_2}$, $K_{\varepsilon,P}$ is a contraction on the set $\mathcal{O} = \mathcal{O}_{C_2} := \{\psi \in X : \|\psi\|_{H^2(D)} \leqq \frac{1}{C_2}\}$.

Observe that, in view of (3.37) and Sobolev's inequality in $H_0^1(D, \mathscr{S}_0)$, one has for all sufficiently large C_2 that

$$\varepsilon\|P\|_{L^4(D)} \leqq C\varepsilon\|P\|_{H^1(D)} \leqq \frac{C}{C_2^{1/2}} < 1 \text{ for all } P \in \mathcal{U}.$$

Estimates (3.19) and (3.21) imply, for $\psi, \tilde{\psi} \in \mathcal{O}$ and $P \in \mathcal{U}$,

$$\|A[\psi] - A[\tilde{\psi}]\|_{L^2(D)} \leqq \frac{C}{C_2}\|\psi - \tilde{\psi}\|_{H^2(D)},$$

$$\|B_\varepsilon[\psi, P] - B_\varepsilon[\tilde{\psi}, P]\|_{L^2(D)} \leqq \frac{C\varepsilon^{-2}}{C_2}\|\psi - \tilde{\psi}\|_{H^2(D)}.$$

Therefore, by Lemma 3.9, we have for $\psi, \tilde{\psi} \in \mathcal{O}$ and $P \in \mathcal{U}$ that

$$\|K_{\varepsilon,P}(\psi) - K_{\varepsilon,P}(\tilde{\psi})\|_{H^2(D)} \leqq \frac{C}{C_2}\|\psi - \tilde{\psi}\|_{H^2(D)}.$$

Also, by (3.18), (3.20) and Lemma 3.9,

$$\begin{aligned}\|K_{\varepsilon,P}(0)\|_{H^2(D)} &\leqq C\varepsilon^2\|B_\varepsilon[0, P]\|_{L^2(D)} \leqq C\varepsilon^2\|P\|_{H^1(D)}\\ &\leqq \frac{C\varepsilon}{C_2^{1/2}} \text{ for all } P \in \mathcal{U}.\end{aligned} \tag{3.38}$$

From the above two estimates, we deduce that there exist a large constant $C_2 > 1$ and a small constant $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and for every $P \in \mathcal{U}$, $K_{\varepsilon,P}$ is a contraction from $\mathcal{O}$ into $\mathcal{O}$ and so has a unique fixed point $\psi_\varepsilon(P) \in \mathcal{O}$.

Proof of (ii) and (iii): We now fix C_2 so that ψ_ε is defined on $\mathcal{U}$ as above and $K_{\varepsilon,P}$ is a contraction from $\mathcal{O}$ into itself.

We will frequently use without explicit reference the estimate below, which is a consequence of (3.37):

$$\varepsilon^2\|P\|_{H^2} \leqq C \text{ for all } P \in \mathcal{U}.$$

It follows from (3.38) and Lemma 3.13 (see below) that the unique fixed point $\psi_\varepsilon(P) \in \mathcal{O}$ of $K_{\varepsilon,P}$ satisfies

$$\|\psi_\varepsilon(P)\|_{H^2(D)} \leqq C\varepsilon^2\|P\|_{H^1(D)},$$

which proves (ii).

Next, Lemma 3.9 and estimate (3.22) imply that

$$\begin{aligned}
&\|K_{\varepsilon,P}(\psi) - K_{\varepsilon,\tilde{P}}(\psi)\|_{H^2(D)} \\
&\quad \leqq C\varepsilon^2 \|\psi\|_{H^2(D)} \left\{ \|P - \tilde{P}\|_{H^2(D)} + (\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)}) \|P - \tilde{P}\|_{L^2(D)} \right\} \\
&\qquad + C\varepsilon^2 \|P - \tilde{P}\|_{H^1(D)} \text{ for all } \psi \in \mathcal{O} \text{ and } P, \tilde{P} \in \mathcal{U}.
\end{aligned}$$

Taking $\psi = \psi_\varepsilon(P)$ and using (ii), we find that

$$\begin{aligned}
&\|\psi_\varepsilon(P) - K_{\varepsilon,\tilde{P}}(\psi_\varepsilon(P))\|_{H^2(D)} \\
&\quad \leqq C\varepsilon^4 \|P\|_{H^1(D)} \left\{ \|P - \tilde{P}\|_{H^2(D)} + (\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)}) \|P - \tilde{P}\|_{L^2(D)} \right\} \\
&\qquad + C\varepsilon^2 \|P - \tilde{P}\|_{H^1(D)} \\
&\quad \leqq C\varepsilon \|P - \tilde{P}\|_\varepsilon \text{ for all } P, \tilde{P} \in \mathcal{U}.
\end{aligned}$$

Applying Lemma 3.13 (see below) to $K_{\varepsilon,\tilde{P}}$ and $b = \psi_\varepsilon(P)$, we obtain

$$\|\psi_\varepsilon(P) - \psi_\varepsilon(\tilde{P})\|_{H^2(D)} \leqq C\varepsilon \|P - \tilde{P}\|_\varepsilon \quad \text{for all } P, \tilde{P} \in \mathcal{U}.$$

This proves (iii) and completes the proof. □

We used the following simple lemma whose proof is omitted.

Lemma 3.13. *If (M, d) is a complete metric space and $K : M \to M$ is a λ-contraction ($0 \leqq \lambda < 1$) with a fixed point $a \in M$, then $d(a, b) \leqq \frac{1}{1-\lambda} d(K(b), b)$ for any $b \in M$.*

3.7. Uniqueness of Critical Points in a Neighborhood of Q_*

In this subsection we show the uniqueness of critical points of $\mathscr{F}_\varepsilon$ in a small neighbourhood of $Q_* \in \{Q_*^\pm\}$ given in (3.6). In particular, we prove the following version of the informal statement (†) formulated in Subsection 3.1:

Proposition 3.14. *For every $C_1 > 0$, there exist large $C_2 > 1$ and small $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leqq \varepsilon_0$, $\mathscr{F}_\varepsilon$ has at most one critical point Q_ε, represented by $(\psi_\varepsilon, P_\varepsilon)$ as in Lemma 3.5, with $\|\psi_\varepsilon\|_{H^2(D)} \leqq \frac{1}{C_2}$, $\varepsilon^2 \|P_\varepsilon\|_{H^2(D)} \leqq \frac{1}{C_2}$, and $\|P_\varepsilon\|_{L^2(D)} \leqq C_1$.*

Remark 3.15. In the proof of Proposition 3.14, the exact form of n_* is used only to have the tubular neighborhood representation (Lemma 3.2) and the stability inequality (Lemma C.2). Therefore, provided these are true, the statement of Proposition 3.14 will hold for more general domains and boundary conditions.

Proof. Let X and Y be defined by (3.25) and (3.30). Let $L_{\varepsilon,\perp}^{-1}$ be as in Definition 3.10 and

$$\theta_\varepsilon := L_{\varepsilon,\perp}^{-1}(s_+ \Delta(n_* \otimes n_*)) \in Y.$$

By Lemma 3.11, as n_* is smooth, we have for every $\varepsilon \in (0, 1)$:

$$\|\theta_\varepsilon\|_\varepsilon \leqq C_0$$

for some constant C_0 independent of ε, and where $\|\cdot\|_\varepsilon$ is as defined in (3.9).

Fix some $C_1 > 0$ and let $\varepsilon_0 \in (0, 1)$ and C_2 be as in Proposition 3.12. By shrinking ε_0 if necessary, we have for $0 < \varepsilon \leqq \varepsilon_0$ that the solution $\psi_\varepsilon(P)$ to (3.15) is defined for all given $P \in \mathcal{U} := \mathcal{U}_{\varepsilon,C_1,C_2}$ (see (3.36)) and

$$\|\psi_\varepsilon(P)\|_{H^2(D)} \leqq C\varepsilon^2 \,\|P\|_{H^1(D)} \leqq C\varepsilon C_2^{-1/2} < 1, \tag{3.39}$$

where we have used (3.37). Here and below, C denotes some constant which may depend on C_1, C_2, a^2, b^2, c^2 but is always independent of ε. For $P \in \mathcal{U}$ we define

$$K_{\varepsilon,\perp}(P) := L_{\varepsilon,\perp}^{-1}\Big(s_+\Delta(n_* \otimes n_*) + \mathring{C}_\varepsilon[\psi_\varepsilon(P), P]\Big) = \theta_\varepsilon + L_{\varepsilon,\perp}^{-1}(\mathring{C}_\varepsilon[\psi_\varepsilon(P), P]),$$

where $\mathring{C}_\varepsilon[\psi, P] = C_\varepsilon[\psi, P] - \frac{1}{3}\mathrm{tr}(C_\varepsilon[\psi, P])I_3$ and C_ε is the operator appearing on the right hand side of (3.16). It should be clear that if P is a fixed point of $K_{\varepsilon,\perp}$, then $(\psi_\varepsilon(P), P)$ solves (3.15)–(3.17), and so the map Q_ε corresponding to $(\psi_\varepsilon(P), P)$ in the representation Lemma 3.2 is a critical point of $\mathscr{F}_\varepsilon$. Therefore, to reach the conclusion, it suffices to show that for all sufficiently small ε, the map $K_{\varepsilon,\perp}$ has at most one fixed point in $\mathcal{U}$. In fact, we show that, for all small ε, $K_{\varepsilon,\perp}$ is contractive on $\mathcal{U}$ with respect to the norm $\|\cdot\|_\varepsilon$.

In the following, we will use Ladyzhenskaya's inequality in two dimensions:

$$\|\varphi\|_{L^4(D)} \leqq C\|\varphi\|_{L^2(D)}^{1/2}\|\nabla\varphi\|_{L^2(D)}^{1/2} \text{ for all } \varphi \in H_0^1(D).$$

In particular, it holds that

$$\|P\|_{L^4(D)} \leqq C\|\nabla P\|_{L^2(D)}^{1/2} \leqq \frac{C}{\varepsilon^{1/2}} \text{ for all } P \in \mathcal{U}. \tag{3.40}$$

Using the estimate (3.23) and inequality (3.40), we have

$$\begin{aligned}
&\|C_\varepsilon[\psi, P] - C_\varepsilon[\tilde\psi, \tilde P]\|_{L^2(D)}\\
&\quad\leqq C\,\|\psi - \tilde\psi\|_{H^2(D)} + C(\|\psi\|_{H^2(D)} + \|\tilde\psi\|_{H^2(D)})\|P - \tilde P\|_{L^2(D)}\\
&\qquad + C\varepsilon^2(\|\nabla P\|_{L^2(D)} + \|\nabla\tilde P\|_{L^2(D)})^{1/2}\|P - \tilde P\|_{H^1(D)}\\
&\quad\leqq C(\|\psi\|_{H^2(D)} + \|\tilde\psi\|_{H^2(D)})\|P - \tilde P\|_{L^2(D)} + C\,\|\psi - \tilde\psi\|_{H^2(D)}\\
&\qquad + C\varepsilon^{\frac{3}{2}}\|P - \tilde P\|_{H^1(D)}
\end{aligned}$$

for all $P, \tilde P \in \mathcal{U}$ and $\psi, \tilde\psi \in X$ with $\|\psi\|_{H^2(D)}, \|\tilde\psi\|_{H^2(D)} \leqq 1$. Thus, by Proposition 3.12(ii) and (iii), we get

$$\begin{aligned}
&\|C_\varepsilon[\psi_\varepsilon(P), P] - C_\varepsilon[\psi_\varepsilon(\tilde P), \tilde P]\|_{L^2(D)}\\
&\quad\leqq C\varepsilon^{1/2}\|P - \tilde P\|_\varepsilon \quad \text{for all } P, \tilde P \in \mathcal{U}.
\end{aligned} \tag{3.41}$$

In view of Lemma 3.11 and (3.41), it follows that

$$\|K_{\varepsilon,\perp}(P) - K_{\varepsilon,\perp}(\tilde P)\|_\varepsilon \leqq C\varepsilon^{\frac{1}{2}}\|P - \tilde P\|_\varepsilon \text{ for all } P, \tilde P \in \mathcal{U}.$$

This implies that, for all sufficiently small ε, $K_{\varepsilon,\perp}$ has at most one fixed point in $\mathcal{U}$, which concludes the proof. □

3.8. Proof of Theorem 3.1

Proof. For $\varepsilon > 0$, let $\mathscr{C}_\varepsilon \subset H^1_{Q_b}(D, \mathscr{S}_0)$ denote the set of minimizers of $\mathscr{F}_\varepsilon$ in $H^1_{Q_b}(D, \mathscr{S}_0)$. Note that if $Q_\varepsilon \in \mathscr{C}_\varepsilon$, then $JQ_\varepsilon J \in \mathscr{C}_\varepsilon$ (where J is given in (1.18)).

Let $n_*^\pm$ be given by (3.5) and $Q_*^\pm = s_+(n_*^\pm \otimes n_*^\pm - \frac{1}{3}I_3)$. It is well known that (see for example [5,13]), if $\varepsilon_m \to 0$ and $Q_{\varepsilon_m} \in \mathscr{C}_{\varepsilon_m}$, then Q_{ε_m} converges along a subsequence in $H^1(D, \mathscr{S}_0)$ to either $Q_* := Q_*^+$ or $Q_*^- = JQ_*^+J$. Thus, with $d = \frac{1}{3}\|Q_*^+ - Q_*^-\|_{H^1(D,\mathscr{S}_0)}$, it holds for all small $\varepsilon > 0$ that

$$\mathscr{C}_\varepsilon = \mathscr{C}_\varepsilon^+ \cup \mathscr{C}_\varepsilon^- \text{ where}$$
$$\mathscr{C}_\varepsilon^\pm := \mathscr{C}_\varepsilon \cap \Big\{Q \in H^1_{Q_b}(D, \mathscr{S}_0) : \|Q_*^\pm - Q\|_{H^1(D,\mathscr{S}_0)} < d\Big\}.$$

It should be clear that $\mathscr{C}_\varepsilon^\pm = J\mathscr{C}_\varepsilon^\mp J$. To conclude, it is enough to show that, for all sufficiently small ε, $\mathscr{C}_\varepsilon^+$ consists of a single map which is $O(2)$-symmetric.

Step 1. We prove that

$$\sup_{Q\in\mathscr{C}_\varepsilon^\pm} \|Q - Q_*^\pm\|_{H^2(D,\mathscr{S}_0)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.42}$$

In fact, it suffices to show that $\sup_{Q\in\mathscr{C}_\varepsilon^+} \|Q - Q_*^+\|_{H^2(D,\mathscr{S}_0)} \to 0$ when $\varepsilon \to 0$. Set $Q_* := Q_*^+$. Arguing indirectly, suppose that there exist $\varepsilon_m \to 0$ and $Q_{\varepsilon_m} \in \mathscr{C}_{\varepsilon_m}^+$ such that $\|Q_{\varepsilon_m} - Q_*\|_{H^2(D,\mathscr{S}_0)} \geqq \frac{1}{C} > 0$. By [32, Theorem 1], Q_{ε_m} converges strongly to Q_* in $C^{1,\sigma}(\bar{D})$ for any $\sigma \in (0, 1)$ and in $C^2_{\mathrm{loc}}(D)$ (note that in the cited paper the results are in three dimensional domains but one can easily check that those convergences also hold in two dimensional domains). Furthermore, by [32, Corollary 2], ΔQ_{ε_m} is bounded in $L^\infty(D)$. By Lebesgue's dominated convergence theorem

$$\lim_{\varepsilon\to 0} \int_D |\Delta Q_{\varepsilon_m}|^2\, \mathrm{d}x = \int_D |\Delta Q_*|^2\, \mathrm{d}x,$$

and so ΔQ_{ε_m} converges to ΔQ_* in $L^2(D, \mathscr{S}_0)$. By elliptic estimates, we conclude that Q_{ε_m} converges to Q_* in $H^2(D, \mathscr{S}_0)$, which gives a contradiction. We have thus established (3.42).

In view of (3.42) and Lemma 3.2, for all sufficiently small ε and $Q_\varepsilon \in \mathscr{C}_\varepsilon^+$, we can represent

$$Q_\varepsilon = \underbrace{s_+\Big(\frac{n_* + \psi_\varepsilon}{|n_* + \psi_\varepsilon|} \otimes \frac{n_* + \psi_\varepsilon}{|n_* + \psi_\varepsilon|} - \frac{1}{3}I_3\Big)}_{=Q_{\varepsilon,\sharp}} + \varepsilon^2 P_\varepsilon,$$

where $\psi_\varepsilon \cdot n_* = 0$ and $P_\varepsilon \in (T_{Q_*}\mathscr{S}_*)^\perp$. We let $\widetilde{\mathscr{C}}_\varepsilon^+$ denote the set of (ψ, P) representing elements of $\mathscr{C}_\varepsilon^+$ as above:

$$\widetilde{\mathscr{C}}_\varepsilon^+ = \Big\{(\psi, P) : s_+\Big(\frac{n_* + \psi}{|n_* + \psi|} \otimes \frac{n_* + \psi}{|n_* + \psi|} - \frac{1}{3}I_3\Big) + \varepsilon^2 P \in \mathscr{C}_\varepsilon^+\Big\}.$$

By (3.42) and Lemma 3.2,

$$\sup_{(\psi,P)\in\widetilde{\mathscr{C}}_\varepsilon^+}\left[\|\psi\|_{H^2(D,\mathbb{R}^3)}+\varepsilon^2\|P\|_{H^2(D,\mathscr{S}_0)}\right]\to 0 \text{ as } \varepsilon\to 0. \tag{3.43}$$

Step 2. In view of (3.43) and Proposition 3.14, in order to prove that $\mathscr{C}_\varepsilon^+$ consists of a single point for all sufficiently small ε, it suffices to show that there exist $\varepsilon_1>0$ and $C_1>0$ such that, for all $\varepsilon\in(0,\varepsilon_1)$,

$$\sup_{(\psi,P)\in\widetilde{\mathscr{C}}_\varepsilon^+}\|P\|_{L^2(D,\mathscr{S}_0)}\leqq C_1. \tag{3.44}$$

We recall some results from [32]. Let $Q_\varepsilon\in\mathscr{C}_\varepsilon^+$ and $(\psi_\varepsilon,P_\varepsilon)\in\widetilde{\mathscr{C}}_\varepsilon^+$ be its corresponding representation as above. We consider the tensor

$$X_\varepsilon:=\frac{1}{\varepsilon^2}\left[Q_\varepsilon^2-\frac{1}{3}s_+Q_\varepsilon-\frac{2}{9}s_+^2I_3\right].$$

(The polynomial on the right hand side is a multiple of the minimal polynomial of matrices belonging to the limit manifold $\mathscr{S}_*$.) By [32, Proposition 4],

$$X_\varepsilon \text{ is bounded in } C^0(\bar D).$$

As $Q_{\varepsilon,\sharp}\in\mathscr{S}_*$, we have $Q_{\varepsilon,\sharp}^2-\frac{1}{3}s_+Q_{\varepsilon,\sharp}-\frac{2}{9}s_+^2I_3=0$ and thus

$$X_\varepsilon=Q_{\varepsilon,\sharp}P_\varepsilon+P_\varepsilon Q_{\varepsilon,\sharp}-\frac{1}{3}s_+P_\varepsilon+\varepsilon^2P_\varepsilon^2. \tag{3.45}$$

Let $End(\mathscr{S}_0)$ be the set of linear endomorphisms of $\mathscr{S}_0$ and define $\mu_\varepsilon:D\to End(\mathscr{S}_0)$ by

$$\mu_\varepsilon(x)(M)=(Q_{\varepsilon,\sharp}(x)-Q_*(x))M+M\,(Q_{\varepsilon,\sharp}(x)-Q_*(x))+\varepsilon^2P_\varepsilon(x)M$$

for all $x\in\bar D$ and $M\in\mathscr{S}_0$. Then (3.45) is equivalent to

$$X_\varepsilon=Q_*P_\varepsilon+P_\varepsilon Q_*-\frac{1}{3}s_+P_\varepsilon+\mu_\varepsilon P_\varepsilon$$

where $\mu_\varepsilon P_\varepsilon$ stands for the map $x\mapsto\mu_\varepsilon(x)(P_\varepsilon(x))$. By definition, $P_\varepsilon\in(T_{Q_*}\mathscr{S}_*)^\perp$ and so by [32, Lemma 2], P_ε commutes with Q_*. It follows that

$$X_\varepsilon=2s_+\left(n_*\otimes n_*-\frac{1}{2}I_3\right)P_\varepsilon+\mu_\varepsilon P_\varepsilon.$$

Note that $(n_*\otimes n_*-\frac{1}{2}I_3)$ has eigenvalues $\pm\frac{1}{2}$ and so is invertible when considered as an endomorphism of $\mathscr{S}_0$ and that $\lim_{\varepsilon\to0}\|\mu_\varepsilon\|_{C^0(\bar D)}=0$ (in view of (3.43) and the embedding $H^2(D)\hookrightarrow C^0(\bar D)$). Thus, as X_ε is bounded in $C^0(\bar D)$, we have that P_ε is also bounded in $C^0(\bar D)$, and in particular in $L^2(D)$. The assertion in (3.44) is established. By Proposition 3.14, we hence have for all sufficiently small ε that $\mathscr{C}_\varepsilon^+$ consists of a single element and $\mathscr{C}_\varepsilon$ consists of exactly two distinct $\mathbb{Z}_2$-conjugate elements.

Step 3: Let Q_ε denote the unique element of $\mathscr{C}_\varepsilon^+$. We show that Q_ε is $O(2)$-symmetric , that is $Q_\varepsilon = (Q_\varepsilon)_{\alpha,\psi}$ for every $\alpha \in \{0, 1\}$ and $\psi \in [0, 2\pi)$, where $(Q_\varepsilon)_{\alpha,\psi}$ is defined as in (1.10).

Indeed, on one hand, it is clear that $(Q_\varepsilon)_{\alpha,\psi} \in H^1_{Q_b}(D, \mathscr{S}_0)$ and

$$\begin{aligned}\|Q_* - (Q_\varepsilon)_{\alpha,\psi}\|_{H^1(D,\mathscr{S}_0)} &= \|(Q_*)_{\alpha,\psi} - (Q_\varepsilon)_{\alpha,\psi}\|_{H^1(D,\mathscr{S}_0)}\\ &= \|Q_* - Q_\varepsilon\|_{H^1(D,\mathscr{S}_0)} < d.\end{aligned}$$

On the other hand, since $(Q_\varepsilon)_{\alpha,\psi}$ has the same energy as Q_ε, it follows $(Q_\varepsilon)_{\alpha,\psi}$ is a minimizer for $\mathscr{F}_\varepsilon$. It follows that $(Q_\varepsilon)_{\alpha,\psi} \in \mathscr{C}_\varepsilon^+$ and so $(Q_\varepsilon)_{\alpha,\psi} = Q_\varepsilon$, as desired.

Step 4: Finally, note that Q_ε and $JQ_\varepsilon J$ are distinct as $\mathscr{C}_\varepsilon^+ \cap \mathscr{C}_\varepsilon^- = \emptyset$, so they are not $\mathbb{Z}_2$-symmetric. This completes the proof. □

3.9. Proof of Theorem 1.5

Proof. Using Theorem 3.1 and a simple scaling argument, we find $R_0 = R_0(a^2, b^2, c^2, k) > 0$ such that for all $R > R_0$, there exist exactly two global minimizers $Q^\pm$ of $\mathscr{F}[\cdot; B_R]$ subjected to the boundary condition (1.7) and these minimizers are k-fold $O(2)$-symmetric and are $\mathbb{Z}_2$-conjugate to each other. By Proposition 1.2, we can express $Q^\pm$ in the form

$$Q^\pm(x) = w_0(|x|)E_0 + w_1(|x|)E_1 \pm w_3(|x|)E_3 \quad \text{for every } x \in B_R.$$

It is clear that $(w_0, w_1, 0, \pm w_3, 0)$ satisfies (2.8)–(2.12).

Now, note that in view of formula (2.4), $\mathscr{F}[Q^\pm; B_R] = \mathscr{F}[w_0E_0 + w_1E_1 \pm |w_3|E_3; B_R]$. Hence, $w_0E_0 + w_1E_1 \pm |w_3|E_3$ are also minimizers of $\mathscr{F}[\cdot; B_R]$ satisfying (1.7). By the above uniqueness up to $\mathbb{Z}_2$-conjugation, we may assume that $w_3 \geqq 0$ in B_R. Also, as $Q^+ \neq Q^-$, $w_3 \not\equiv 0$. Recalling equation (2.11) and noting that $w_2 = w_4 = 0$, we can apply the strong maximum principle to conclude that $w_3 > 0$ in $(0, R)$.[14] The proof is complete. □

4. Mountain Pass Critical Points

In this section, we give the proof of Theorem 1.6, which asserts the existence of at least five $O(2)$-symmetric critical points satisfying the boundary condition (1.7) for $\mathscr{F}^R := \mathscr{F}[\cdot; B_R]$ for all large enough R.

We denote by $\mathscr{A}_R^{rs}$ and $\mathscr{A}_R^{str}$ the sets of k-fold $O(2)$-symmetric and $\mathbb{Z}_2 \times O(2)$-symmetric maps, respectively, satisfying the boundary conditions (1.7):

$$\mathscr{A}_R^{rs} = \Big\{Q \in H^1_{Q_b}(B_R, \mathscr{S}_0) : Q \text{ is } O(2)\text{-symmetric}\Big\},$$
$$\mathscr{A}_R^{str} = \Big\{Q \in H^1_{Q_b}(B_R, \mathscr{S}_0) : Q \text{ is } \mathbb{Z}_2 \times O(2)\text{-symmetric}\Big\}.$$

---

[14] Alternatively, if w_3 was zero somewhere, its derivative would be zero there and the uniqueness for ODE would imply $w_3 \equiv 0$, which is not possible.

By the characterization of symmetric maps (see Propositions 1.2 and 2.9), we can express the sets $\mathscr{A}_R^{rs}$ and $\mathscr{A}_R^{str}$ in terms of the basis components defined in Section 2 as follows:

$$\mathscr{A}_R^{rs} = \Big\{ Q(x) = w_0(|x|)E_0 + w_1(|x|)E_1 + w_3(|x|)E_3 \ : \\ w_0 \in H^1((0,R); r\,\mathrm{d}r),\, w_1, w_3 \in H^1((0,R); r\,\mathrm{d}r) \cap L^2\Big((0,R); \frac{1}{r}\,\mathrm{d}r\Big), \\ w_0(R) = -\frac{s_+}{\sqrt{6}},\, w_1(R) = \frac{s_+}{\sqrt{2}},\, w_3(R) = 0 \Big\},$$

$$\mathscr{A}_R^{str} = \Big\{ Q(x) = w_0(|x|)E_0 + w_1(|x|)E_1 \ : \\ w_0 \in H^1((0,R); r\,\mathrm{d}r),\, w_1 \in H^1((0,R); r\,\mathrm{d}r) \cap L^2\Big((0,R); \frac{1}{r}\,\mathrm{d}r\Big), \\ w_0(R) = -\frac{s_+}{\sqrt{6}},\, w_1(R) = \frac{s_+}{\sqrt{2}} \Big\}.$$

A direct computation shows that critical points of $\mathscr{F}^R$ in $\mathscr{A}_R^{rs}$ or $\mathscr{A}_R^{str}$ are in fact critical points of $\mathscr{F}^R$ in $H^1_{Q_b}(B_R, \mathscr{S}_0)$ (cf. Remark 2.4). To prove Theorem 1.6, we use the fact that $\mathscr{F}^R$ has two global minimizers in $\mathscr{A}_R^{rs}$ (due to Theorem 1.5) and the mountain pass theorem. An energetic consideration is needed to show that the obtained mountain pass critical point does not coincide with critical points of $\mathscr{F}^R$ in $\mathscr{A}_R^{str}$.

We start with an estimate for the minimal energy of $\mathscr{F}^R$ in $\mathscr{A}_R^{str}$.

Lemma 4.1. *There exists some $C > 0$ depending only on a^2, b^2 and c^2 such that, for all $\delta \in (0,1)$, $k \in \mathbb{Z}\backslash\{0\}$ and $R > \max(1, \frac{Cek^2}{\delta^2})$, there holds*

$$\frac{\pi s_+^2 k^2}{2} \ln R + Ck^2 \geqq \alpha_R := \min_{\mathscr{A}_R^{str}} \mathscr{F}^R \\ \geqq \frac{\pi s_+^2 k^2}{2(1+2\delta)^2} \left(\ln \frac{\delta^2 R}{Ck^2} - \ln\ln \frac{\delta^2 R}{Ck^2} \right). \tag{4.1}$$

As a consequence,

$$\lim_{R\to\infty} \frac{\alpha_R}{\ln R} = \frac{1}{2}\pi s_+^2 k^2. \tag{4.2}$$

Remark 4.2. In [4], it was shown that $\mathscr{F}^R$ has critical points whose energies are of order $k \ln R$; and these are not $SO(2)$-symmetric for $k \neq \pm 1$.

Proof. By (2.4), we have

$$\alpha_R = 2\pi \min \Big\{ \mathscr{E}_R[w_0, w_1] : w_0 \in H^1((0,R); r\,\mathrm{d}r), \\ w_1 \in H^1((0,R); r\,\mathrm{d}r) \cap L^2\Big((0,R); \frac{1}{r} dr\Big), \\ w_0(R) = -\frac{s_+}{\sqrt{6}},\, w_1(R) = \frac{s_+}{\sqrt{2}} \Big\},$$

where

$$\mathscr{E}_R[w_0, w_1] = \int_0^R \left\{ \frac{1}{2}[|w_0'|^2 + |w_1'|^2] + \frac{k^2}{2r^2}|w_1|^2 + h(w_1, w_0) \right\} r\,dr,$$
$$h(x, y) = \left(-\frac{a^2}{2} + \frac{c^2}{4}[|x|^2 + |y|^2] \right) [|x|^2 + |y|^2] - \frac{b^2\sqrt{6}}{18} y(y^2 - 3x^2) - f_*,$$

and f_* is given by (1.3).

We note (see for example [24, Lemma 5.1]) that $h(x, y) \geqq 0$ and equality holds if and only if (x, y) belongs to the set $\{(\pm\frac{s_+}{\sqrt{2}}, -\frac{s_+}{\sqrt{6}}), (0, \frac{2s_+}{\sqrt{6}})\}$. Furthermore, the Hessian of h is positive definite at these critical points. In particular, one has

$$h(x, y) \geqq \frac{1}{C}\left(x - \frac{s_+}{\sqrt{2}}\right)^2 \text{ for all } (x, y) \text{ satisfying}$$
$$x^2 + y^2 \leqq \frac{2}{3}s_+^2, \frac{s_+}{3\sqrt{2}} \leqq x \leqq \frac{s_+}{\sqrt{2}}, \tag{4.3}$$

where, here and below, C denotes some positive constant (that may change from line to line) which depends only on a^2, b^2 and c^2, and in particular is always independent of R, k and δ.
Step 1: Proof of the upper bound for α_R in (4.1).

Consider the test function $(\bar{w}_0, \bar{w}_1)$ defined by $\bar{w}_0(r) \equiv -\frac{s_+}{\sqrt{6}}$ and $\bar{w}_1(r) = \frac{s_+}{\sqrt{2}} \min(r, 1)$. Then

$$\alpha_R \leqq 2\pi \mathscr{E}_R[\bar{w}_0, \bar{w}_1] = 2\pi \mathscr{E}_1[\bar{w}_0, \bar{w}_1] + 2\pi \int_1^R \frac{s_+^2 k^2}{4r} dr \leqq Ck^2 + \frac{1}{2}\pi s_+^2 k^2 \ln R,$$

which provides the upper bound on α_R, given in the left hand side of (4.1).
Step 2: Proof of the lower bound for α_R in (4.1).

Let (w_0, w_1) be a minimizer of $\mathscr{E}_R$ the existence of which is guaranteed by the direct method of the calculus of variations. We fix some $\delta \in (0, 1)$. Due to the fact that w_1 is continuous, $w_1(0) = 0$ (see Step 4 of the proof of [23, Proposition 2.3]) and $w_1(R) = \frac{s_+}{\sqrt{2}}$, there exists the largest number $R_1 \in (0, R)$ such that $w_1(R_1) = \frac{s_+}{(1+2\delta)\sqrt{2}}$. By the same arguments, there exists the smallest number $R_2 \in (R_1, R)$ such that $w_1(R_2) = \frac{s_+}{(1+\delta)\sqrt{2}}$.

As $w_1(r) \geqq \frac{s_+}{(1+2\delta)\sqrt{2}}$ in $[R_1, R]$, we have

$$\frac{1}{2\pi}\alpha_R = \mathscr{E}_R(w_0, w_1) \geqq \int_{R_1}^R \frac{k^2}{2r}|w_1|^2\, \mathrm{d}r \geqq \frac{s_+^2 k^2}{4(1+2\delta)^2} \ln \frac{R}{R_1}.$$

It follows that

$$R_1 \geqq R \exp\left(-\frac{2(1+2\delta)^2 \alpha_R}{\pi s_+^2 k^2} \right). \tag{4.4}$$

On the other hand, by the definition of R_1 and R_2, we have $\frac{s_+}{(1+2\delta)\sqrt{2}} \leqq w_1 \leqq \frac{s_+}{(1+\delta)\sqrt{2}}$ in $[R_1, R_2]$. Also, by [23, Eq. (3.12)], $w_0^2 + w_1^2 \leqq \frac{2}{3}s_+^2$. Thus, using (4.3), we have

$$h(w_1, w_0) \geqq \frac{1}{C}\left(w_1 - \frac{s_+}{\sqrt{2}}\right)^2 > \frac{\delta^2}{C} \text{ in } [R_1, R_2].$$

Therefore, it follows that

$$\frac{1}{2\pi}\alpha_R = \mathscr{E}_R(w_0, w_1) \geqq \int_{R_1}^{R_2} h(w_0, w_1)\, r\, \mathrm{d}r \geqq \frac{\delta^2}{C}(R_2^2 - R_1^2) \geqq \frac{\delta^2}{C} R_1(R_2 - R_1),$$

and hence, in view of (4.4),

$$\frac{R_2 - R_1}{R_1} \leqq \frac{C\alpha_R}{\delta^2 R^2} \exp\left(\frac{4(1+2\delta)^2\alpha_R}{\pi s_+^2 k^2}\right).$$

This leads, by Cauchy-Schwarz' inequality, to

$$\begin{aligned}\frac{1}{\pi}\alpha_R &= 2\mathscr{E}_R(w_0, w_1) \geqq R_1 \int_{R_1}^{R_2} |w_1'|^2\, \mathrm{d}r \geqq \frac{R_1}{R_2 - R_1}\left(\int_{R_1}^{R_2} w_1'\, \mathrm{d}r\right)^2 \\ &= \frac{\delta^2 s_+^2}{2(1+\delta)^2(1+2\delta)^2}\frac{R_1}{R_2 - R_1} \geqq \frac{\delta^4 R^2}{C\alpha_R}\exp\left(-\frac{4(1+2\delta)^2\alpha_R}{\pi s_+^2 k^2}\right).\end{aligned}$$

Rearranging, we obtain $\Lambda e^{\Lambda} \geqq \frac{\delta^2 R}{Ck^2}$ for $\Lambda := \frac{2(1+2\delta)^2\alpha_R}{\pi s_+^2 k^2}$, which implies

$$\frac{2(1+2\delta)^2\alpha_R}{\pi s_+^2 k^2} = \Lambda \geqq \ln\frac{\delta^2 R}{Ck^2} - \ln\ln\frac{\delta^2 R}{Ck^2} \quad \text{provided} \quad \ln\frac{\delta^2 R}{Ck^2} \geqq 1.$$

The conclusion of the result is immediate. □

In order to use the mountain pass theorem to show that $\mathscr{F}^R$ has more than two critical points, we need to exhibit a path γ connecting the two minimizers $Q_R^{\pm}$ of $\mathscr{F}^R$ such that

$$\sup_t \mathscr{F}^R[\gamma(t)] < \alpha_R$$

where α_R is the minimal energy of $\mathscr{F}^R|_{\mathscr{A}_R^{str}}$.

The existence of such a path is a priori not clear. Indeed, note that, as $R \to \infty$ and after a suitable rescaling, $Q_R^{\pm}$ tend to $Q_*^{\pm}$ (see (3.6)). As maps from D into $\mathscr{S}_*$, Q_*^+ and Q_*^- belong to different homotopy classes and so cannot be connected by a continuous path in $H^1(D, \mathscr{S}_*)$. The desired path γ must therefore necessarily leave the limit manifold $\mathscr{S}_*$. In particular, the contribution of the bulk energy potential f_{bulk} to $\mathscr{F}^R[\gamma(t)]$ cannot be neglected.

We construct the path γ by exploiting the conformal invariance of the Dirichlet energy in two dimensions to connect $Q_R^{\pm}$ to $Q_{R_0}^{\pm}$ for some fixed R_0 by using $Q_r^{\pm}$ (with variable r) and their inverted copies, and then finally connect $Q_{R_0}^+$ and $Q_{R_0}^-$. As a result we obtain a mountain pass path with energy $O_k(1)$ (see (4.5)), which is clearly less than α_R for large R.

Proof of Theorem 1.6. In the proof, C denotes some positive constant which is always independent of R. As denoted earlier, critical points of $\mathscr{F}^R$ in $\mathscr{A}_R^{rs}$ or $\mathscr{A}_R^{str}$ are critical points of $\mathscr{F}^R$ in $H^1_{Q_b}(B_R, \mathscr{S}_0)$. Therefore it suffices to work with $\mathscr{F}^R\big|_{\mathscr{A}_R^{rs}}$. To simplify the notation in what follows we still use $\mathscr{F}^R$ instead of $\mathscr{F}^R\big|_{\mathscr{A}_R^{rs}}$.

By Theorem 1.5, there exists $R_0 > 0$ such that, for $R \geqq R_0$, $\mathscr{F}^R$ has two distinct minimizers in $\mathscr{A}_R^{rs}$ which are $O(2)$-symmetric but not $\mathbb{Z}_2 \times O(2)$-symmetric (in fact, they are $\mathbb{Z}_2$-conjugate). We label these minimizers as $Q_R^{\pm}$ and claim that, for any $0 < d < \|Q_R^+ - Q_R^-\|_{H^1(B_R)}$, we have

$$\inf \left\{ \mathscr{F}^R[Q] : Q \in \mathscr{A}_R^{rs}, \|Q - Q_R^+\|_{H^1(B_R)} = d \right\} > \mathscr{F}^R[Q_R^{\pm}].$$

Assume by contradiction that there exists a sequence $\{Q_m\}_{m\in\mathbb{N}} \subset \mathscr{A}_R^{rs}$ satisfying $\|Q_m - Q_R^+\|_{H^1(B_R)} = d$ such that $\mathscr{F}^R[Q_m] \to \mathscr{F}^R[Q_R^{\pm}]$ as $m \to \infty$. Without loss of generality, we can also assume that Q_m is weakly convergent in $H^1(B_R, \mathscr{S}_0)$ and strongly convergent in $L^p(B_R, \mathscr{S}_0)$ for any $p \in [1, \infty)$. The limit of Q_m is then a minimizer of $\mathscr{F}^R$, and thus, by Theorem 1.5 and our assumption on d, must coincide with Q_R^+. Now, as $Q_m \to Q_R^+$ in $L^4(B_R, \mathscr{S}_0)$ and $\mathscr{F}^R[Q_m] \to \mathscr{F}^R[Q_R^+]$, we have that $\|\nabla Q_m\|_{L^2(B_R)} \to \|\nabla Q_R^+\|_{L^2(B_R)}$, which further implies that $Q_m \to Q_R^+$ in $H^1(B_R, \mathscr{S}_0)$ as $m \to \infty$. This contradicts the fact that $\|Q_m - Q_R^+\|_{H^1(B_R)} = d > 0$ for every m. The claim is proved.

We proceed to check that $\mathscr{F}^R$ satisfies the Palais-Smale condition in the sense of [34, Definition 12.3] (with the closed convex set $\mathscr{A}_R^{rs}$ in the space $H^1(B_R, \mathscr{S}_0)$). Indeed, let $(Q_m) \subset \mathscr{A}_R^{rs}$ be a Palais-Smale sequence for $\mathscr{F}^R$, that is $\mathscr{F}_R[Q_m]$ is bounded and

$$\sup_{P\in\mathscr{A}_R^{rs}, \|P-Q_m\|_{H^1}\leqq 1} \langle Q_m - P, D\mathscr{F}^R(Q_m)\rangle \to 0 \text{ as } m \to \infty.$$

As $f_{\text{bulk}} \geqq 0$, Q_m is bounded in $H^1(B_R, \mathscr{S}_0)$. Now note that $D\mathscr{F}^R(Q_m) = -\Delta Q_m + V(Q_m)$ for some (polynomially) nonlinear operator $V : H^1(B_R, \mathscr{S}_0) \to H^{-1}(B_R, \mathscr{S}_0)$ which, by the compact embedding theorem, maps bounded sets of $H^1(B_R, \mathscr{S}_0)$ into relatively compact sets of $H^{-1}(B_R, \mathscr{S}_0)$. Thus, up to extracting a subsequence, we may assume that $Q_m \rightharpoonup Q$ weakly in H^1 and $V(Q_m) \to V(Q)$ in H^{-1} and in L^p for any $p \in (1, \infty)$. Hence $D\mathscr{F}^R(Q_m)$ converges weakly in H^{-1} to $D\mathscr{F}^R(Q)$. Fix some arbitrary $P \in \mathscr{A}_R^{rs}$. Being a Palais-Smale sequence, (Q_m) satisfies

$$|\langle Q - P, D\mathscr{F}^R(Q_m)\rangle| \to 0.$$

It follows from these two statements that

$$\int_{B_R} \big[|\nabla Q|^2 - \nabla P : \nabla Q + V(Q)(Q - P)\big]\,\mathrm{d}x = \langle Q - P, D\mathscr{F}^R(Q)\rangle = 0.$$

Returning to (Q_m) being a Palais-Smale sequence, we have

$$\int_{B_R} \big[|\nabla Q_m|^2 - \nabla P : \nabla Q_m + V(Q_m)(Q_m - P)\big]\,\mathrm{d}x$$

$$= \langle Q_m - P, D\mathscr{F}^R(Q_m)\rangle \to 0.$$

Combining the last two equations, the weak convergence of Q_m to Q in H^1 and the strong convergence of $V(Q_m)$ to $V(Q)$ in L^2, we deduce that

$$\int_{B_R} |\nabla Q_m|^2 \,\mathrm{d}x \to \int_{B_R} |\nabla Q|^2 \,\mathrm{d}x.$$

We conclude that Q_m converges to Q in H^1 and that the Palais-Smale condition holds.

Applying the mountain pass theorem [34, Theorem 12.8], we conclude for $R \geqq R_0$ that $\mathscr{F}^R$ has a mountain pass critical point in $\mathscr{A}_R^{rs}$ at minimax energy level associated to paths connecting $Q_R^{\pm}$, which will be denoted by Q_R^{mp}.

Let us prove now that $Q_R^{mp} \notin \mathscr{A}_R^{str}$ for sufficiently large R. We use the following lemma, which will be proved later.

Lemma 4.3. *There exists some $C > 0$ independent of k such that*

$$\mathscr{F}^R[Q_R^{mp}] \leqq C(R_0^2 + |k|) \textit{ for all } R > R_0. \tag{4.5}$$

Furthermore there exists a continuous path $\gamma : [-2, 2] \to \mathscr{A}_R^{rs}$ such that $\gamma(\pm 2) = Q_R^{\pm}$ and

$$\mathscr{F}^R[\gamma(t)] \leqq C(R_0^2 + |k|) \textit{ for all } t \in [-2, 2], \tag{4.6}$$

where C is independent of R, k and t.

To proceed, take $R_1 > \max(R_0, 4C_1 e k^2)$ such that, for all $R > R_1$ and $k \in 2\mathbb{Z}\backslash\{0\}$, we have

$$\frac{\pi s_+^2 k^2}{8}\left(\ln\frac{R}{4C_1k^2} - \ln\ln\frac{R}{4C_1k^2}\right) > C_2(R_0^2 + |k|),$$

where C_1 is the constant from (4.1) corresponding to $\delta = 1/2$, and the constants C_2 and R_0 are the ones from (4.5). Then the mountain pass critical point Q_R^{mp} (which belongs to $\mathscr{A}_R^{rs}$) has the energy $\mathscr{F}^R(Q_R^{mp})$ bounded from above by $C_2(R_0^2 + |k|)$ and thus does not belong to $\mathscr{A}_R^{str}$ thanks to Lemma 4.1. In other words, Q_R^{mp} is $O(2)$-symmetric and is not $\mathbb{Z}_2 \times O(2)$-symmetric.

Let us now construct a second mountain pass critical point $\tilde{Q}_R^{mp}$. Indeed, the lack of $\mathbb{Z}_2 \times O(2)$-symmetry implies that in the decomposition $Q_R^{mp}(x) = w_0(|x|)E_0 + w_1(|x|)E_1 + w_3(|x|)E_3$ we have that $w_3 \not\equiv 0$. It follows that $\tilde{Q}_R^{mp}(x) = w_0(|x|)E_0 + w_1(|x|)E_1 - w_3(|x|)E_3$ is an additional critical point of $\mathscr{F}^R$ with the same energy as Q_R^{mp} (it is necessarily of mountain pass type).

Let us now construct a fifth critical point Q_R^{str} that will be k-radially symmetric. Indeed, minimizing the energy $\mathscr{F}^R\big|_{\mathscr{A}_R^{str}}$ one can show that $\mathscr{F}^R$ has a critical point in $\mathscr{A}_R^{str}$, called Q_R^{str}. By the above energy estimates and Lemma 4.1, it is clear that Q_R^{str} differs from $Q_R^{\pm}$, Q_R^{mp} and $\tilde{Q}_R^{mp}$. We have thus shown that $\mathscr{F}^R$ has at least five k-fold $O(2)$-symmetric critical points in $\mathscr{A}_R$, at least four of which are not k-fold $\mathbb{Z}_2 \times O(2)$-symmetric. □

To finish, we provide the

Proof of Lemma 4.3. Let $n_*^\pm$ be defined by (3.5). Its rescaled version to B_R is given by

$$n_{R,*}^\pm(r\cos\varphi, r\sin\varphi) = \Big(\frac{2R^{\frac{k}{2}}r^{\frac{k}{2}}\cos(\frac{k}{2}\varphi)}{R^k + r^k}, \frac{2R^{\frac{k}{2}}r^{\frac{k}{2}}\sin(\frac{k}{2}\varphi)}{R^k + r^k}, \pm\frac{R^k - r^k}{R^k + r^k}\Big).$$

We define $Q_{R,*}^\pm = s_+(n_{R,*}^\pm \otimes n_{R,*}^\pm - \frac{1}{3}I_3)$ and note that $f_{\text{bulk}}(Q_{R,*}^\pm) \equiv 0$. It follows that

$$\begin{aligned} \mathscr{F}^R[Q_R^\pm] \leqq \mathscr{F}^R[Q_{R,*}^\pm] &= \frac{1}{2}\int_{B_R} |\nabla Q_{R,*}^\pm|^2\,\mathrm{d}x \\ &= s_+^2 \int_D |\nabla n_*^\pm|^2 = 4\pi |k|\, s_+^2. \end{aligned} \tag{4.7}$$

Step 1. We first construct $\gamma\big|_{[-2,-1]\cup[1,2]}$. For that, let $r_1, r_2 : [-2, 2] \to [R_0, R]$ be given by

$$r_1(t) = \begin{cases} R_0, & t \in [-1, 1], \\ (R - R_0)|t| + 2R_0 - R, & t \in [-2, 2]\backslash[-1, 1], \end{cases}$$
$$r_2(t) = (r_1(t)R)^{1/2}.$$

For $1 \leqq t \leqq 2$ we define $\gamma(\pm t) : B_R \to \mathscr{S}_0$ by

$$\gamma(\pm t)(x) = \begin{cases} Q_{r_1(t)}^\pm(x) & \text{if } |x| \leqq r_1(t), \\ Q_R^+(\frac{r_2(t)^2}{|x|^2}x) & \text{if } r_1(t) < |x| < r_2(t), \\ Q_R^+(x) & \text{if } r_2(t) \leqq |x| \leqq R. \end{cases}$$

To dispel confusion, we note that on the lower two cases (that is, $r_1(t) < |x| < R$), we are using the "plus" minimizing branch Q_R^+. Since for any $r > 0$ we have $Q_r^\pm(r\frac{x}{|x|}) = s_+(n \otimes n - \frac{1}{3}I_3)$, the inner and outer traces of $\gamma(t)$ at $\partial B_{r_1(t)}$ coincide and so $\gamma(t)$ belongs to $\mathscr{A}_R^{rs}$; see Fig. 2. The continuity of γ with respect to t is a consequence of a priori L^∞ estimates [31, Proposition 3], elliptic estimates and the uniqueness part of Theorem 1.5.

By construction, it is clear that $\gamma(\pm 2) = Q_R^\pm$ and $\gamma(\pm 1)\big|_{B_{R_0}} = Q_{R_0}^\pm$.

Let us check that (4.6) holds for $1 \leqq |t| \leqq 2$. In view of (4.7) and the fact that the integrand of $\mathscr{F}$ is non-negative, we have

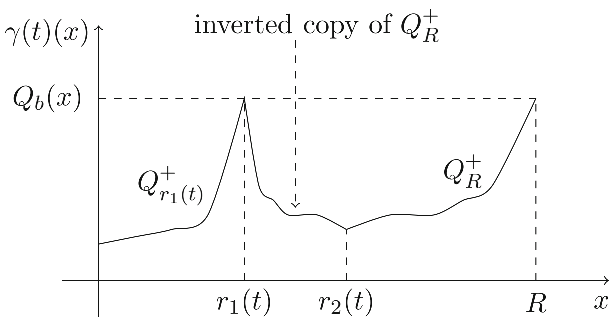

Fig. 2. A schematic 'graph' of $\gamma(t)$ for $t \in [1, 2]$

$$\mathscr{F}^{r_1(t)}[Q^{\pm}_{r_1(t)}] \leqq C|k|, \quad \mathscr{F}[Q^+_R, B_R \backslash B_{r_2(t)}] \leqq \mathscr{F}^R[Q^+_R] \leqq C|k|.$$

Therefore, we only need to show that

$$\int_{B_{r_2(t)} \backslash B_{r_1(t)}} \left[\frac{1}{2}|\nabla\gamma(t)|^2 + f_{\text{bulk}}(\gamma(t))\right] \mathrm{d}x \leqq 4\pi|k|s_+^2. \tag{4.8}$$

Indeed, by a change of variable $y = \frac{|r_2(t)|^2}{|x|^2}x$, we have in view of (4.7):

$$\begin{aligned}
&\int_{B_{r_2(t)} \backslash B_{r_1(t)}} \left[\frac{1}{2}|\nabla\gamma(t)|^2 + f_{\text{bulk}}(\gamma(t))\right] \mathrm{d}x \\
&\quad = \int_{B_R \backslash B_{r_2(t)}} \left[\frac{1}{2}|\nabla Q^+_R|^2 + \underbrace{\frac{r_2(t)^4}{|y|^4}}_{\leqq 1} f_{\text{bulk}}(Q^+_R)\right] \mathrm{d}y \\
&\quad \leqq \mathscr{F}^R[Q^+_R] \leqq 4\pi|k|s_+^2,
\end{aligned}$$

which proves (4.8).

Step 2. We continue the argument by letting $\gamma\big|_{(-1,1)}$ be the linear interpolation between $\gamma(\pm 1)$, that is $\gamma(t) = \frac{1}{2}[(t+1)\gamma(1) - (t-1)\gamma(-1)]$.

We now check (4.6) for $|t| \leqq 1$. Note that $\gamma(t) = \frac{1}{2}[(t+1)Q^+_{R_0} - (t-1)Q^-_{R_0}]$ in B_{R_0}. A standard argument using the maximum principle (see the proof of [31, Proposition 3]) shows that $|Q^{\pm}_{R_0}| \leqq \sqrt{\frac{2}{3}}s_+$, which yields $|\gamma(t)| \leqq \sqrt{\frac{2}{3}}s_+$ in B_{R_0}. Hence

$$\mathscr{F}^{R_0}[\gamma(t)] \leqq CR_0^2 + \frac{1}{2}\int_{B_{R_0}} |\nabla\gamma(t)|^2 \,\mathrm{d}x.$$

This together with the convexity of the Dirichlet energy, the non-negativity of the integrand of $\mathscr{F}$, and (4.7) gives

$$\mathscr{F}^{R_0}[\gamma(t)] \leqq C(R_0^2 + |k|).$$

On the other hand, as $\gamma(t)(x) = \gamma(1)(x)$ in $B_R \backslash B_{R_0}$, we have, in view of non-negativity of the integrand of $\mathscr{F}$,

$$\mathscr{F}^R[\gamma(t)] = \mathscr{F}^{R_0}[\gamma(t)] + \mathscr{F}[\gamma(1), B_R \backslash B_{R_0}] \leqq \mathscr{F}^{R_0}[\gamma(t)] + \mathscr{F}^R[\gamma(1)].$$

Recalling (4.6) for $t = 1$, we conclude the proof of the claim. □

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A Remarks on Minimizers for Odd k

In this appendix we would like to make some remarks on the energy bounds and the symmetry properties of minimizers of the energy

$$\mathscr{F}_\varepsilon[Q;\Omega] := \int_\Omega \Big[\frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f_{\mathrm{bulk}}(Q)\Big]\,\mathrm{d}x$$

defined on a fixed finitely-connected, bounded C^1 domain Ω and subject to a given C^1 planar $\mathscr{S}_*$-valued boundary condition Q_b, that is Q_b takes values in the set

$$\mathscr{S}_*^{planar} = \Big\{s_+\Big(v\otimes v - \frac{1}{3}I_3\Big) \,:\, v\in\mathbb{S}^1\Big\}$$
$$\text{with } \mathbb{S}^1 = \{(v_1, v_2, 0)\in\mathbb{R}^3 \,:\, |v| = 1\}\subset\mathbb{R}^3. \tag{A.1}$$

Note that $\mathscr{S}_*^{planar}$ is homeomorphic to $\mathbb{R}P^1$ and continuous maps from $\partial\Omega$ into $\mathscr{S}_*^{planar}$ have well-defined $\frac{1}{2}\mathbb{Z}$-valued degrees; see for instance Brezis, Coron and Lieb [8, Section VIII, part B]. The relation between $\mathscr{S}_*^{planar}$ and $\mathbb{S}^1$ in (A.1) is manifested in the fact that a map $Q\in C(\partial\Omega,\mathscr{S}_*^{planar})$ can be written in the form $Q = s_+\big(v\otimes v - \frac{1}{3}I_3\big)$ for some $v\in C(\partial\Omega,\mathbb{S}^1)$ if and only if the degree $\frac{k}{2}$ of Q is an integer (that is, $k\in\mathbb{Z}$ is even), and, in which case, is equal to that of v.

In the discussion to follow, we assume that Q_b has degree $\frac{k}{2}$ with $k\in\mathbb{Z}\backslash\{0\}$ (which does not necessarily have the form (1.8)–(1.9)). We consider both cases of **even and odd** k and note that they are fundamentally different in the limit $\varepsilon\to 0$: for odd k the limiting energy is infinite, while for even k the limit has finite energy.

As usual, let $H^1_{Q_b}(\Omega,\mathscr{S}_0)$ denote the set of H^1 maps from Ω into $\mathscr{S}_0$ equal to Q_b on $\partial\Omega$.

Remark A.1. Let $\Omega \subset \mathbb{R}^2$ be a fixed finitely-connected, bounded C^1 domain and $Q_b : \partial\Omega \to \mathscr{S}_*^{planar}$ be an arbitrary C^1 map of degree $\frac{k}{2}$ for some $k \in \mathbb{Z}$.

For **even** $k \neq 0$, there exist $\varepsilon_0 > 0$ and $C_1, C_2 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$C_1 \leqq \min_{H^1_{Q_b}(\Omega,\mathscr{S}_0)} \mathscr{F}_\varepsilon[\cdot;\Omega] \leqq C_2. \tag{A.2}$$

For **odd** k, there exist $\varepsilon_1 > 0$ and $C > 0$ such that for all $\varepsilon < \varepsilon_1$

$$\frac{\pi}{2} s_+^2 |\ln\varepsilon| - C \leqq \min_{H^1_{Q_b}(\Omega,\mathscr{S}_0)} \mathscr{F}_\varepsilon[\cdot;\Omega] \leqq \frac{\pi}{2} s_+^2 |\ln\varepsilon| + C. \tag{A.3}$$

Remark A.2. For the unit disk $\Omega = D$, the following can be stated in the case the boundary data Q_b is given by (1.8)–(1.9). We have seen that when k is **even** the minimizers of $\mathscr{F}_\varepsilon$ in $H^1_{Q_b}(D,\mathscr{S}_0)$ are k-fold $O(2)$-symmetric.

When k is **odd** the symmetry of minimizers is more delicate and so far is **unknown**. For $k = \pm 1$, we conjecture that there exists a unique minimizer and this minimizer is $\mathbb{Z}_2 \times O(2)$-symmetric—see [24,25] for some supporting evidence. In the case of odd $|k| > 1$ the above energy bounds forbid minimizers to have $SO(2)$-symmetry (in view of Lemma 4.1 and [23, Propositions 2.1 and 2.3]) as well as the configuration of k-vortices of degree $\pm\frac{1}{2}$ constructed in [4].

We would like now to briefly outline how one can obtain energy bounds (A.2) and (A.3). When D is a disk and k is odd, these bounds were established by CANEVARI [10,11]. We will see below that (A.2) and the upper bound in (A.3) can be established by elementary arguments using some extension property for $\mathscr{S}_*^{planar}$-valued maps of degree zero. The proof of the lower bound in (A.3) is more substantial and draws on the corresponding aforementioned estimate for disks [11].

Let us start with (A.2) when k is even. The lower bound $C_1 > 0$ can be taken to be the minimal Dirichlet energy under the given (non-constant) boundary data Q_b (since $f_{\text{bulk}} \geqq 0$). For the upper bound, we construct a test function by splitting the domain Ω into a disk $B_R \subset \Omega$, which without loss of generality is assumed to be centered at the origin, of some small radius R, and the complement $\Omega\backslash B_R$. On the boundary of the disk, we impose the test function to have boundary data of the form (1.8)–(1.9). We define Q_{test} by joining together minimizers of $\mathscr{F}_\varepsilon[\cdot; B_R]$ and $\mathscr{F}_\varepsilon[\cdot;\Omega\backslash B_R]$ with respect to the indicated boundary data on each subdomain. It is clear that the minimal energy satisfies the following bound:

$$\begin{aligned}\min_{H^1_{Q_b}(\Omega,\mathscr{S}_0)} \mathscr{F}_\varepsilon[\cdot;\Omega] &\leqq \mathscr{F}_\varepsilon[Q_{test};\Omega]\\ &= \min \mathscr{F}_\varepsilon[Q; B_R] + \min \mathscr{F}_\varepsilon[Q;\Omega\backslash B_R],\end{aligned} \tag{A.4}$$

where the two minimizations on the right hand side are under the constraint that $Q = Q_{test}$ on the respected boundary. We know (see Table 1) that the first term on the right hand side of (A.4) is bounded uniformly in ε. Since the degree of the map $Q_{test} : \partial(\Omega\backslash B_R) \to \mathscr{S}_*^{planar}$ is zero we can use an H^1 extension—see Lemma A.3 below—to show that the second term on the right hand side of (A.4) is bounded uniformly in ε. Estimate (A.2) follows.

Lemma A.3. *Let $G \subset \mathbb{R}^2$ be a bounded finitely-connected C^1 domain. Then every map $Q \in C^1(\partial G, \mathscr{S}_*^{planar})$ of degree zero extends to a map $Q \in H^1(G, \mathscr{S}_*^{planar})$.*

Proof. As $\mathscr{S}_*^{planar}$ is diffeomorphic to a circle, this result is well known (see [6, Section I.2]). See also [19, p.126] for results on continuous extensions in any dimension. $\square$

We now consider the upper bound in (A.3) when k is odd. We select two disjoint disks $B_R(x_1)$ and $B_R(x_2)$ inside Ω. On the boundary of these disks, we consider the test function Q_{test}:

$$Q_{test}(x) = s_+\Big(n(x) \otimes n(x) - \frac{1}{3} I_3\Big), \qquad x \in \partial B_R(x_1) \cup \partial B_R(x_2), \quad \text{(A.5)}$$

where we set for $Re^{i\varphi} = (R\cos\varphi, R\sin\varphi) \in \partial B_R(0), 0 \leqq \varphi < 2\pi$:

$$\begin{aligned} n(x_1 + Re^{i\varphi}) &:= \left(\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2}, 0\right), \\ n(x_2 + Re^{i\varphi}) &:= \left(\cos\frac{(k-1)\varphi}{2}, \sin\frac{(k-1)\varphi}{2}, 0\right). \end{aligned}$$

In $\Omega \backslash (B_R(x_1) \cup B_R(x_2))$ and $B_R(x_2)$, the test function Q_{test} is constructed by minimizing $\mathscr{F}_\varepsilon$ under the indicated boundary conditions. In $B_R(x_1)$, Q_{test} is a minimizer of $\mathscr{F}_\varepsilon$ in $\mathbb{Z}_2 \times O(2)$-symmetry. Using Lemma 4.1 (namely the upper bound in (4.1)) and arguing as in the previous case, we arrive at the upper bound in (A.3).

We turn to the lower bound in (A.3) when k is odd. Take some large disk $B_{R'}(0) \supset \Omega$. We impose on $\partial B_{R'}(0)$ a boundary condition of the form (A.5) where

$$n(R'\cos\varphi, R'\sin\varphi) = \left(\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2}, 0\right), \quad 0 \leqq \varphi < 2\pi.$$

By [11, Proposition 15], we have within the above boundary condition (B.C.) on $\partial B_{R'}(0)$:

$$\min_{B.C.} \mathscr{F}_\varepsilon[\cdot; B_{R'}(0)] \geqq \frac{\pi}{2} s_+^2 |\ln\varepsilon| - C,$$

and by (A.2), within the above boundary condition (B.C.) on $\partial(B_{R'}(0)\backslash\Omega)$:

$$\min_{B.C.} \mathscr{F}_\varepsilon[\cdot; B_{R'}(0)\backslash\Omega] \leqq C.$$

The lower bound in (A.3) follows from the above two estimates and the inequality

$$\min_{B.C.} \mathscr{F}_\varepsilon[\cdot; B_{R'}(0)] \leqq \min_{H^1_{Q_b}(\Omega, \mathscr{S}_0)} \mathscr{F}_\varepsilon[\cdot; \Omega] + \min_{B.C.} \mathscr{F}_\varepsilon[\cdot; B_{R'}(0)\backslash\Omega].$$

B The Euler–Lagrange Equations Near Q_*

In this appendix, we give the proof of Lemma 3.5 and Proposition 3.6 which concern the Euler–Lagrange equations for critical points of $\mathscr{F}$ relative to the representation in Lemma 3.2.

Proof of Lemma 3.5. We set $v = \frac{n_*+\psi}{|n_*+\psi|}$, $\hat{v} = n_* + \psi$. Recall that $n_* \cdot \psi = 0$, $Q_\sharp = s_+(v \otimes v - \frac{1}{3}I_3)$ and $Q = Q_\sharp + \varepsilon^2 P$. We calculate separately the elastic part in $\mathscr{F}_\varepsilon$ and then the bulk term.

1. The elastic part:

$$\int_\Omega |\nabla Q|^2 \,\mathrm{d}x = \int_\Omega \Big[2s_+^2 \,|\nabla v|^2 + 2\varepsilon^2 s_+ \nabla(v \otimes v) : \nabla P + \varepsilon^4 |\nabla P|^2\Big] \,\mathrm{d}x. \tag{B.1}$$

(Here $\nabla X : \nabla Y = \sum_i (\nabla_i X \cdot \nabla_i Y)$ for two matrix-valued maps X and Y.)

We calculate individually the first two terms.

a. The $|\nabla v|^2$ term. Using the identities

$$\nabla v = \frac{1}{|\hat{v}|}\nabla \hat{v} - \frac{1}{2|\hat{v}|^3}\hat{v} \otimes \nabla |\hat{v}|^2,$$

and $n_* \cdot \psi = 0$ (in particular $|\hat{v}|^2 = 1 + |\psi|^2$), we see that

$$\begin{aligned} |\nabla v|^2 &= \frac{1}{|\hat{v}|^2}(|\nabla \hat{v}|^2 - \frac{1}{4|\hat{v}|^2}|\nabla |\hat{v}|^2|^2) \\ &= \frac{1}{1+|\psi|^2}\Big(|\nabla \psi|^2 + |\nabla n_*|^2 + 2\nabla n_* \cdot \nabla \psi - \frac{1}{4(1+|\psi|^2)}|\nabla |\psi|^2|^2\Big). \end{aligned}$$

Noting that

$$\begin{aligned} &\int_D \frac{1}{1+|\psi|^2}\nabla n_* \cdot \nabla \psi \,\mathrm{d}x \\ &\quad = -\int_D \frac{1}{1+|\psi|^2}\Big[\Delta n_* - \frac{1}{1+|\psi|^2}\nabla n_* \cdot \nabla |\psi|^2\Big] \cdot \psi \,\mathrm{d}x \\ &\quad = \int_D \frac{1}{(1+|\psi|^2)^2}(\nabla n_* \cdot \nabla |\psi|^2) \cdot \psi \,\mathrm{d}x, \end{aligned}$$

we obtain

$$\begin{aligned} \int_D |\nabla v|^2 \,\mathrm{d}x = \int_D |\nabla n_*|^2 \,\mathrm{d}x + \int_D \big[|\nabla \psi|^2 - |\nabla n_*|^2 \,|\psi|^2\big] \,\mathrm{d}x \\ + \int_D g(x, \psi, \nabla \psi) \,\mathrm{d}x, \end{aligned} \tag{B.2}$$

where g is super-cubic (meaning at least cubic) in $(\psi, \nabla\psi)$ at zero:

$$g(x,\psi,\nabla\psi) = \frac{1}{1+|\psi|^2}\Big[-|\psi|^2(|\nabla\psi|^2 - |\nabla n_*|^2\,|\psi|^2) \\ + \frac{2}{1+|\psi|^2}(\nabla n_* \cdot \nabla|\psi|^2)\cdot\psi - \frac{1}{4(1+|\psi|^2)}|\nabla|\psi|^2|^2\Big].$$

b. The gradient coupling term $\nabla(v\otimes v) : \nabla P$. We write

$$\begin{aligned}
&\int_\Omega \nabla(v\otimes v) : \nabla P \\
&\quad= \int_\Omega \nabla\Big(\frac{1}{1+|\psi|^2}(n_*+\psi)\otimes(n_*+\psi)\Big) : \nabla P \\
&\quad= \int_\Omega \nabla(n_*\otimes n_*) : \nabla P + \nabla(n_*\otimes\psi + \psi\otimes n_*) : \nabla P + \nabla\hat{g}(x,\psi) : \nabla P
\end{aligned} \tag{B.3}$$

where $\hat{g}$ is super-quadratic in ψ at $\psi = 0$:

$$\hat{g}(x,\psi) = -\frac{|\psi|^2}{1+|\psi|^2}(n_*+\psi)\otimes(n_*+\psi) + \psi\otimes\psi.$$

The expression $\nabla(n_*\otimes\psi + \psi\otimes n_*) : \nabla P$ on the right hand side of (B.3) contains some terms which are quadratic in the derivatives of P and ψ. However, we can eliminate this quadratic character by using some specific geometric information as follows: we note that $P \in (T_{Q_*}\mathscr{S}_*)^\perp$ and $n_*\otimes\psi + \psi\otimes n_* \in T_{Q_*}\mathscr{S}_*$. Indeed, by [32, Lemma 2], Pn_* is parallel to n_* and so $Pn_* = (P\cdot(n_*\otimes n_*))n_*$. Also, as $\psi\cdot n_* = 0$ and $\Delta n_* \parallel n_*$, we have $\Delta\psi\cdot n_* = -2\nabla\psi\cdot\nabla n_*$. Thus, integrating by parts using again $\Delta n_* \parallel n_*$ gives

$$\begin{aligned}
&\int_D \nabla(n_*\otimes\psi) : \nabla P\,\mathrm{d}x \\
&\quad= \int_D \sum_k \nabla_k P\cdot(\nabla_k n_*\otimes\psi + n_*\otimes\nabla_k\psi)\,\mathrm{d}x \\
&\quad= \int_D \Big[\sum_k \nabla_k P\cdot(\nabla_k n_*\otimes\psi) - (Pn_*)\cdot\Delta\psi - ((\nabla n_*)^t\,P)\cdot\nabla\psi\Big]\mathrm{d}x \\
&\quad= \int_D \Big[\sum_k \nabla_k P\cdot(\nabla_k n_*\otimes\psi) \\
&\qquad + 2(P\cdot(n_*\otimes n_*))(\nabla n_*\cdot\nabla\psi) - ((\nabla n_*)^t\,P)\cdot\nabla\psi\Big]\mathrm{d}x.
\end{aligned}$$

As P is symmetric, we hence get

$$\int_D \nabla(n_*\otimes\psi + \psi\otimes n_*) : \nabla P\,\mathrm{d}x$$

$$
\begin{aligned}
&= 2\int_D \Big[\sum_k \nabla_k P \cdot (\nabla_k n_* \otimes \psi) \\
&\quad + 2(P \cdot (n_* \otimes n_*))(\nabla n_* \cdot \nabla \psi) - ((\nabla n_*)^t \, P) \cdot \nabla \psi \Big] \,\mathrm{d}x.
\end{aligned} \tag{B.4}
$$

It is readily seen that the integral on the right hand side is linear in the derivatives of P and ψ. This cancellation will play a role in our analysis: its contribution to the Euler–Lagrange equations of $\mathscr{F}_\varepsilon$ is of first order rather than second order.

2. The bulk part: We expand $f_{\text{bulk}}(Q) = f_{\text{bulk}}(Q_\sharp + \varepsilon^2 P)$ in terms of powers of ε. As $\mathscr{S}_*$ is the set of minimum points of f_{bulk}, we have $f_{\text{bulk}}(Q_\sharp) = 0$ and $\nabla f_{\text{bulk}}(Q_\sharp) = 0$. We have:

$$
\begin{aligned}
f_{\text{bulk}}(Q) &= f_{\text{bulk}}(Q) - f_{\text{bulk}}(Q_\sharp) - \varepsilon^2 \nabla f_{\text{bulk}}(Q_\sharp) \cdot P \\
&= \varepsilon^4 \left[-\frac{a^2}{2}|P|^2 - b^2 P^2 \cdot Q_\sharp + \frac{c^2}{2}|Q_\sharp|^2 |P|^2 + c^2 |P \cdot Q_\sharp|^2 \right] \\
&\quad + \varepsilon^6 \left[-\frac{b^2}{3}\operatorname{tr}(P^3) + c^2 P \cdot Q_\sharp \, |P|^2 \right] + \varepsilon^8 \frac{c^2}{4}|P|^4 \\
&= \varepsilon^4 \, h(x, P) + \varepsilon^4 \hat{h}(x, \psi, P) + \varepsilon^6 \, \mathring{h}_\varepsilon(x, \psi, P),
\end{aligned} \tag{B.5}
$$

where:

$$
\begin{aligned}
h(x, P) &= \frac{b^2 \, s_+}{2}|P|^2 - b^2 \, s_+ P^2 \cdot (n_* \otimes n_*) + c^2 \, s_+^2 |P \cdot (n_* \otimes n_*)|^2 \\
\hat{h}(x, \psi, P) &= -b^2 \, s_+ P^2 \cdot (v \otimes v - n_* \otimes n_*) \\
&\quad + c^2 \, s_+^2 [|P \cdot (v \otimes v)|^2 - |P \cdot (n_* \otimes n_*)|^2], \\
\mathring{h}_\varepsilon(x, \psi, P) &= -\frac{b^2}{3}\operatorname{tr}(P^3) + c^2 s_+ P \cdot (v \otimes v) \, |P|^2 + \varepsilon^2 \frac{c^2}{4}|P|^4.
\end{aligned}
$$

(Here we have used the identity $-a^2 - \frac{b^2}{3}s_+ + \frac{2c^2}{3}s_+^2 = 0$.) Note also that, as $n_* \cdot \psi = 0$ and $P n_* \parallel n_*$, $\hat{h}(x, \psi, P)$ and $\mathring{h}_\varepsilon(x, \psi, P) - \mathring{h}_\varepsilon(x, 0, P)$ are super-quadratic in ψ at $\psi = 0$.

We now put together all the previous expressions, to get a new form of the full energy. Using the expression of $|\nabla v|^2$ from (B.2) in (B.1) and putting (B.4) in (B.3) and then in (B.1) provide the elastic part. Putting this together with the expansion of the bulk part (B.5) into (3.1) we obtain

$$
\begin{aligned}
\mathscr{F}_\varepsilon[Q] &= s_+^2 \int_D |\nabla n_*|^2 \,\mathrm{d}x + s_+^2 \int_D \big[|\nabla \psi|^2 - |\nabla n_*|^2 \, |\psi|^2 \big] \,\mathrm{d}x \\
&\quad + \varepsilon^2 \int_D \Big[\frac{\varepsilon^2}{2}|\nabla P|^2 + h(x, P) \Big] \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon^2 s_+ \int_D \Big[\nabla(n_* \otimes n_*) : \nabla P + 2 \sum_k \nabla_k P : (\nabla_k n_* \otimes \psi) \\
&+ 4(P \cdot (n_* \otimes n_*))(\nabla n_* \cdot \nabla \psi) - 2((\nabla n_*)^t \, P) \cdot \nabla \psi \Big] \, \mathrm{d}x \\
&+ \int_D [s_+^2 g(x, \psi, \nabla \psi) + \varepsilon^2 \, \hat{h}(x, \psi, P)] \, \mathrm{d}x \\
&+ \varepsilon^2 \int_D \Big[s_+ \nabla \hat{g}(x, \psi) : \nabla P + \varepsilon^2 \mathring{h}_\varepsilon(x, \psi, P)) \Big] \, \mathrm{d}x.
\end{aligned}
$$

The Euler–Lagrange equations for $\mathscr{F}_\varepsilon$ in terms of ψ and P are then readily found to be of the form

$$
-\Delta \psi - |\nabla n_*|^2 \, \psi = A[\psi] + \varepsilon^2 \, B_\varepsilon[\psi, P] + \lambda_\varepsilon(x) \, n_*, \tag{B.6}
$$

$$
-\varepsilon^2 \Delta P + \mathring{\nabla}_P h(x, P) = s_+ \Delta(n_* \otimes n_*) + C_\varepsilon[\psi, P] - \frac{1}{3} \mathrm{tr}(C_\varepsilon[\psi, P]) I_3 + F_\varepsilon(x), \tag{B.7}
$$

where

$$
\begin{aligned}
\mathring{\nabla}_P h(x, P) &:= \nabla_P h(x, P) - \frac{1}{3} \mathrm{tr}(\nabla_P h(x, P)) I_3 \\
&= b^2 \, s_+ \, P + \frac{2}{s_+^2} (-b^2 \, s_+ + c^2 \, s_+^2) \, (P \cdot Q_*) \, Q_*
\end{aligned} \tag{B.8}
$$

is the gradient of h with respect to $P \in \mathscr{S}_0$,[15] λ_ε is a Lagrange multiplier accounting for the constraint $\psi \cdot n_* = 0$, $F_\varepsilon(x) \in T_{Q_*} \mathscr{S}_*$ is a Lagrange multiplier accounting for the constraint $P(x) \in (T_{Q_*} \mathscr{S}_*)^\perp$, and

$$
A[\psi]_j = \frac{1}{2} \nabla_i \Big[\frac{\partial g}{\partial(\nabla_i \psi_j)}(x, \psi, \nabla \psi) \Big] - \frac{1}{2} \frac{\partial g}{\partial \psi_j}(x, \psi, \nabla \psi), \tag{B.9}
$$

$$
\begin{aligned}
B_\varepsilon[\psi, P]_j &= -\frac{1}{2 s_+^2} \frac{\partial \hat{h}}{\partial \psi_j}(x, \psi, P) - \frac{1}{2 s_+^2} \varepsilon^2 \frac{\partial \mathring{h}_\varepsilon}{\partial \psi_j}(x, \psi, P) \\
&\quad + \frac{1}{2 s_+} \frac{\partial \hat{g}}{\partial \psi_j}(x, \psi) \cdot \Delta P \\
&\quad + \frac{1}{s_+} \nabla_i \big[2(P \cdot (n_* \otimes n_*)) \nabla_i (n_*)_j - \nabla_i (n_*)_k \, P_{kj} \big] \\
&\quad - \frac{1}{s_+} \nabla_i P_{jk} \, \nabla_i (n_*)_k,
\end{aligned} \tag{B.10}
$$

$$
C_\varepsilon[\psi, P]_{ij} = -\frac{\partial \hat{h}}{\partial P_{ij}}(x, \psi, P) - \varepsilon^2 \frac{\partial \mathring{h}_\varepsilon}{\partial P_{ij}}(x, \psi, P) + s_+ \Delta \hat{g}_{ij}(x, \psi)
$$

[15] In deriving (B.8), it is useful to keep in mind the relation that $P n_* \parallel n_*$.

$$+\ 2s_+\nabla_k(\nabla_k(n_*)_i\psi_j) - 4s_+(n_*)_i(n_*)_j(\nabla n_*\cdot\nabla\psi) \\ +\ 2s_+\nabla_k(n_*)_i\nabla_k\psi_j. \tag{B.11}$$

This finishes the proof of Lemma 3.5. □

We continue with the proof of the Lipschitz-type estimates for A, B_ε and C_ε:

Proof of Proposition 3.6. Using the definitions of $A[\psi]$, $B_\varepsilon[\psi, P]$ given in (B.9), (B.10) together with the fact that $\frac{\partial}{\partial\psi_j}\hat{h}(x,0,P) = \frac{\partial}{\partial\psi_j}\mathring{h}_\varepsilon(x,0,P) = 0$,[16] we obtain $A[0]=0$,

$$\begin{aligned}
|A[\psi]-A[\tilde\psi]| &\leqq C(|\psi|+|\tilde\psi|) \\
&\qquad \Big[(1+|\Delta\psi|+|\Delta\tilde\psi|)|\psi-\tilde\psi| + |\nabla^2(\psi-\tilde\psi)| \\
&\qquad +\ (1+|\nabla\psi|+|\nabla\tilde\psi|)|\nabla(\psi-\tilde\psi)|\Big] \\
&\qquad +\ C(|\nabla\psi|+|\nabla\tilde\psi|)(1+|\nabla\psi|+|\nabla\tilde\psi|)|\psi-\tilde\psi|, \\
|B_\varepsilon(0,P)| &\leqq C(|\nabla P|+|P|), \\
|B_\varepsilon[\psi,P]-B_\varepsilon[\tilde\psi,P]| &\leqq C\,(|\Delta P|+|P|^2+\varepsilon^2|P|^3)|\psi-\tilde\psi|, \\
|B_\varepsilon[\psi,P]-B_\varepsilon[\psi,\tilde P]| &\leqq C|\psi|[|\Delta(P-\tilde P)|+(|P|+|\tilde P|) \\
&\qquad (1+\varepsilon^2(|P|+|\tilde P|))|P-\tilde P|] \\
&\qquad +\ C[|\nabla(P-\tilde P)|+|P-\tilde P|],
\end{aligned}$$

which imply the claimed estimates (3.18), (3.19), (3.20), (3.21) and (3.22) in view of the embedding $H^2(D)\hookrightarrow W^{1,4}(D)\hookrightarrow L^\infty(D)$.

We split $C_\varepsilon[\psi,P] = C_\varepsilon^{(1)}[\psi,P] + C^{(2)}[\psi]$ where

$$\begin{aligned}
C_\varepsilon^{(1)}[\psi,P]_{ij} &= -\frac{\partial\hat h}{\partial P_{ij}}(x,\psi,P) - \varepsilon^2\frac{\partial\mathring{h}_\varepsilon}{\partial P_{ij}}(x,\psi,P), \\
C^{(2)}[\psi]_{ij} &= 2s_+\nabla_k(\nabla_k(n_*)_i\psi_j) - 4s_+(n_*)_i(n_*)_j(\nabla n_*\cdot\nabla\psi) \\
&\qquad + 2s_+\nabla_k(n_*)_i\nabla_k\psi_j + s_+\Delta\hat g_{ij}(x,\psi).
\end{aligned}$$

We have

$$\begin{aligned}
|C^{(2)}[\psi]-C^{(2)}[\tilde\psi]| &\leqq C(1+|\Delta\psi|+|\Delta\tilde\psi|+|\nabla\psi|^2+|\nabla\tilde\psi|^2)|\psi-\tilde\psi| \\
&\qquad +\ C(1+|\psi|+|\tilde\psi|+|\nabla\psi|+|\nabla\tilde\psi|)|\nabla\psi-\nabla\tilde\psi| \\
&\qquad +\ C(|\psi|+|\tilde\psi|)|\Delta\psi-\Delta\tilde\psi|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|C^{(2)}[\psi]-C^{(2)}[\tilde\psi]\|_{L^2(D)} &\leqq C(1+\|\psi\|_{H^2(D)}+\|\tilde\psi\|_{H^2(D)}+\|\nabla\psi\|^2_{L^4(D)} \\
&\qquad +\|\nabla\tilde\psi\|^2_{L^4(D)})\|\psi-\tilde\psi\|_{H^2(D)}.
\end{aligned}$$

---

[16] Recall that this is a consequence of the relations $n_*\cdot\psi=0$ and $Pn_*\parallel n_*$.

As for $C_\varepsilon^{(1)}$, we have

$$
\begin{aligned}
&|C_\varepsilon^{(1)}[\psi, P] - C_\varepsilon^{(1)}[\tilde{\psi}, \tilde{P}]| \\
&\quad \leqq C\Big[|\psi| + |\tilde{\psi}| + \varepsilon^2(|P| + |\tilde{P}|)\big(1 + \varepsilon^2(|P| + |\tilde{P}|)\big)\Big]|P - \tilde{P}| \\
&\qquad + C(|P| + |\tilde{P}|)\big(1 + \varepsilon^2(|P| + |\tilde{P}|)\big)\,|\psi - \tilde{\psi}|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|C_\varepsilon^{(1)}[\psi, P] - C_\varepsilon^{(1)}[\tilde{\psi}, \tilde{P}]\|_{L^2(D)} \\
&\quad \leqq C(\|\psi\|_{H^2(D)} + \|\tilde{\psi}\|_{H^2(D)})\|P - \tilde{P}\|_{L^2(D)} \\
&\qquad + C(\|P\|_{L^2(D)} + \|\tilde{P}\|_{L^2(D)}) \\
&\qquad \big(1 + \varepsilon^2(\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)})\big)\|\psi - \tilde{\psi}\|_{H^2(D)} \\
&\qquad + C\varepsilon^2(\|P\|_{L^4(D)} + \|\tilde{P}\|_{L^4(D)}) \\
&\qquad \big(1 + \varepsilon^2(\|P\|_{H^2(D)} + \|\tilde{P}\|_{H^2(D)})\big)\|P - \tilde{P}\|_{H^1(D)},
\end{aligned}
$$

and thus we obtain the claimed estimate (3.23). □

C Proof of Proposition 3.7

Proposition 3.7 easily follows from Lax-Milgram's theorem and Lemma C.2 below.

Lemma C.1. *Let n_* be given by (3.5). For any $\zeta \in H_0^1(D, \mathbb{R})$, there holds*

$$
I[\zeta] := \int_D [|\nabla \zeta|^2 - |\nabla n_*|^2\,\zeta^2]\,\mathrm{d}x \geqq 0.
$$

In particular, n_* is a stable harmonic map. Equality holds if and only if $\zeta = \frac{t(1-r^k)}{1+r^k}$ for some $t \in \mathbb{R}$.

Proof. Let $L_\| = -\Delta - |\nabla n_*|^2$. W.l.o.g., we assume $n_* := n_*^+$. Then $n_3 = n_* \cdot e_3 = \frac{1-r^k}{1+r^k} > 0$ in D and note that $L_\| n_3 = 0$. Decomposing $\zeta = n_3 \xi$, a direct computation yields (cf. [22, Lemma A.1])

$$
I[\zeta] := \int_D [|\nabla \zeta|^2 - |\nabla n_*|^2\,\zeta^2]\,\mathrm{d}x = \int_D n_3^2 |\nabla \xi|^2\,\mathrm{d}x \geqq 0.
$$

The assertion follows. □

Lemma C.2. *Let n_* be given by (3.5). Then n_* is strictly stable, that is there exists some number $c_0 > 0$ such that for any $\zeta \in H_0^1(D, \mathbb{R}^3)$ with $\zeta \cdot n_* = 0$ almost everywhere in D, there holds*

$$I[\zeta] = \int_D [|\nabla\zeta|^2 - |\nabla n_*|^2\,|\zeta|^2]\,\mathrm{d}x \geqq c_0 \int_D [|\nabla\zeta|^2 + |\zeta|^2]\,\mathrm{d}x.$$

Proof. By Lemma C.1, I is non-negative on $H_0^1(D,\mathbb{R}^3)$. Let

$$\lambda_1 = \inf\Big\{ \; I[\zeta] : \zeta \in H_0^1(D,\mathbb{R}^3),\ \|\zeta\|_{L^2(D)} = 1,\ \zeta\cdot n_* = 0 \text{ almost everywhere in } D\Big\} \geqq 0.$$

Using the boundedness of $|\nabla n_*|$, we can apply the direct method of the calculus of variations to show that λ_1 is achieved by some $\bar\zeta \in H_0^1(D,\mathbb{R}^3)$ satisfying $\|\bar\zeta\|_{L^2(D)} = 1$ and $\bar\zeta\cdot n_* = 0$ almost everywhere in D.

If $\lambda_1 = 0$, Lemma C.1 implies that each component of $\bar\zeta$ is proportional to $\frac{1-r^k}{1+r^k}$. However, as $\bar\zeta\cdot n_* = 0$, this is possible only if $\bar\zeta \equiv 0$, which contradicts $\|\bar\zeta\|_{L^2(D)} = 1$.

We thus have that $\lambda_1 > 0$. Consequently, as $|\nabla n_*|$ is bounded, there exists some $\eta > 0$ such that, for any $\zeta \in H_0^1(D,\mathbb{R}^3)$ with $\zeta\cdot n_* = 0$ almost everywhere in D,

$$\int_D [|\nabla\zeta|^2 - |\nabla n_*|^2\,|\zeta|^2]\,\mathrm{d}x \geqq \lambda_1 \int_D |\zeta|^2\,\mathrm{d}x \geqq \eta \int_D |\nabla n_*|^2\,|\zeta|^2\,\mathrm{d}x,$$

which implies

$$\int_D [|\nabla\zeta|^2 - |\nabla n_*|^2\,|\zeta|^2]\,\mathrm{d}x \geqq \frac{\eta}{1+\eta}\int_D |\nabla\zeta|^2\,\mathrm{d}x.$$

The conclusion is readily seen. □

D A Calculus Lemma

Lemma D.1. *Let $g(x,y,z) = 2x^3 - 6xy^2 + 3xz^2 + 3\sqrt{3}yz^2$ for every $x,y,z\in\mathbb{R}$. Then*

$$-2(x^2+y^2+z^2)^{3/2} \leqq g(x,y,z) \leqq 2(x^2+y^2+z^2)^{3/2}.$$

Equality in the first inequality holds if and only if $(x,y,z) = s(\frac{1}{2},\frac{\sqrt{3}}{2},0)$ for some $s \geqq 0$ or $x + \sqrt{3}y = -\sqrt{x^2+y^2+z^2}$. Equality in the second inequality holds if and only if $(x,y,z) = s(\frac{1}{2},\frac{\sqrt{3}}{2},0)$ for some $s \leqq 0$ or $x+\sqrt{3}y = \sqrt{x^2+y^2+z^2}$.

Proof. Since g is three-homogeneous, it suffices to consider the extremization problem

$$\max\{g : x^2+y^2+z^2 = 1\} \text{ and } \min\{g : x^2+y^2+z^2=1\}.$$

We rewrite $g = (x + \sqrt{3}y)(2x^2 + 3z^2 - 2\sqrt{3}xy)$, and so when $x^2 + y^2 + z^2 = 1$,

$$g = (x + \sqrt{3}y)(3 - (x + \sqrt{3}y)^2) = \tilde{g}(x + \sqrt{3}y),$$

where $\tilde{g}(t) = 3t - t^3$. As $|x + \sqrt{3}y| \leqq 2$ when $x^2 + y^2 + z^2 = 1$, $\max_{[-2,2]} \tilde{g} = 2$, which is achieved for $t \in \{-2, 1\}$, $\min_{[-2,2]} \tilde{g} = -2$, which is achieved for $t \in \{-1, 2\}$, the conclusion follows. $\square$

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Radu Ignat
Institut de Mathématiques de Toulouse, Institut Universitaire de France, UMR 5219,
Université de Toulouse, CNRS, UPS IMT,
31062 Toulouse Cedex 9
France.
e-mail: Radu.Ignat@math.univ-toulouse.fr

and

Luc Nguyen
Mathematical Institute and St Edmund Hall,
University of Oxford,
Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road,
Oxford
OX2 6GG UK.
e-mail: luc.nguyen@maths.ox.ac.uk

and

Valeriy Slastikov
School of Mathematics,
University of Bristol,
University Walk,
Bristol
BS8 1TW UK.
e-mail: Valeriy.Slastikov@bristol.ac.uk

and

Arghir Zarnescu
IKERBASQUE, Basque Foundation for Science,
Maria Diaz de Haro 3,
48013 Bilbao
Bizkaia
Spain.

and

BCAM, Basque Center for Applied Mathematics,
Mazarredo 14,
48009 Bilbao
Bizkaia
Spain.

and

"Simion Stoilow" Institute of the Romanian Academy,
21 Calea Griviţei,
010702 Bucharest
Romania.
e-mail: azarnescu@bcamath.org

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